

MATH 3705B
Test 4 Solutions
March 17, 2011

[Marks] Questions 1-2 are multiple choice. Circle the correct answer. Only the answer will be marked.

- [4] 1. The solution of the wave equation $u_{xx} = \frac{1}{9}u_{tt}$, $0 < x < 2$, which satisfies the boundary conditions $u(0, t) = u(2, t) = 0$, is given by

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right) \left[a_n \cos\left(\frac{3n\pi t}{2}\right) + b_n \sin\left(\frac{3n\pi t}{2}\right) \right].$$

If $u(x, t)$ satisfies the initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = 3 \sin(\pi x) - \sin(3\pi x)$, then the coefficients a_n and b_n are given by

- (a) $b_2 = 3$, $b_6 = -1$, $b_n = 0$ otherwise, and $a_n = 0$ for all $n \geq 1$.
- (b) $b_2 = \frac{1}{\pi}$, $b_6 = -\frac{1}{9\pi}$, $b_n = 0$ otherwise, and $a_n = 0$ for all $n \geq 1$.
- (c) $a_2 = -3$, $a_6 = 1$, $a_n = 0$ otherwise, and $b_n = 0$ for all $n \geq 1$.
- (d) $a_2 = -\frac{1}{\pi}$, $a_6 = \frac{1}{9\pi}$, $a_n = 0$ otherwise, and $b_n = 0$ for all $n \geq 1$.
- (e) None of the above

Answer: (b)

- [4] 2. The solution of Laplace's equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ inside the circle $r = 3$ has the form

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

The solution which satisfies the boundary condition $u(3, \theta) = 1 + 2 \sin(3\theta) - 3 \cos(2\theta)$ is

- (a) $u(r, \theta) = 1 - 27r^{-2} \cos(2\theta) + 54r^{-3} \sin(3\theta)$
- (b) $u(r, \theta) = 1 - 3r^2 \cos(2\theta) + 2r^3 \sin(3\theta)$
- (c) $u(r, \theta) = 2 - \frac{1}{3}r^2 \cos(2\theta) + \frac{2}{27}r^3 \sin(3\theta)$
- (d) $u(r, \theta) = 1 - \frac{1}{3}r^2 \cos(2\theta) + \frac{2}{27}r^3 \sin(3\theta)$
- (e) None of the above

Answer: (d)

- [6] 3. Find the polynomial solution $u(x, y) = \alpha x + \beta y + \gamma xy + \delta$ of Laplace's equation $u_{xx} + u_{yy} = 0$ within the rectangle $0 < x < 2$, $0 < y < 1$, which satisfies the boundary conditions

$$u(0, y) = y, \quad u(2, y) = 0, \quad u(x, 0) = 0, \quad u(x, 1) = 1 - \frac{1}{2}x.$$

Write down the complete solution $u(x, y)$.

Solution:

$$u(x, 0) = 0 \Rightarrow \alpha x + \delta = 0 \Rightarrow \alpha = \delta = 0 \Rightarrow u(x, y) = \beta y + \gamma xy.$$

$$u(2, y) = 0 \Rightarrow \beta y + 2\gamma y = 0 \Rightarrow (\beta + 2\gamma)y = 0 \Rightarrow \beta = -2\gamma \Rightarrow u(x, y) = -2\gamma y + \gamma xy.$$

$$u(0, y) = y \Rightarrow -2\gamma y = y \Rightarrow \gamma = -\frac{1}{2} \Rightarrow u(x, y) = y - \frac{1}{2}xy.$$

$$u(x, 1) = 1 - \frac{x}{2} \Rightarrow 1 - \frac{x}{2} = 1 - \frac{x}{2}.$$

$$\text{Thus, } u(x, y) = y - \frac{1}{2}xy.$$

- [8] 4. The solution of Laplace's equation $u_{xx} + u_{yy} = 0$, $0 < x < L$, $0 < y < M$, satisfying the boundary conditions $u(x, 0) = 0$, $u(x, M) = 0$, $u(0, y) = 0$, $u(L, y) = f(y)$, has the form

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi x}{M}\right) \sin\left(\frac{n\pi y}{M}\right).$$

Find the solution of Laplace's equation $u_{xx} + u_{yy} = 0$ within the rectangle $0 < x < 2$, $0 < y < 1$, which satisfies the boundary conditions $u(x, 0) = 0$, $u(x, 1) = 0$, $u(0, y) = 0$, $u(2, y) = 1$. Write down the complete solution $u(x, y)$.

Solution:

$$L = 2, \quad M = 1 \Rightarrow u(x, y) = \sum_{n=1}^{\infty} a_n \sinh(n\pi x) \sin(n\pi y).$$

$$1 = u(2, y) = \sum_{n=1}^{\infty} a_n \sinh(2n\pi) \sin(n\pi y) \Rightarrow$$

$$a_n \sinh(2n\pi) = \frac{2}{1} \int_0^1 \sin(n\pi y) dy = -\frac{2}{n\pi} \cos(n\pi y) \Big|_0^1 = \frac{2}{n\pi} [1 - (-1)^n] \Rightarrow$$

$$a_n = \frac{2[1 - (-1)^n]}{n\pi \sinh(2n\pi)} \Rightarrow u(x, y) = \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n\pi \sinh(2n\pi)} \sinh(n\pi x) \sin(n\pi y).$$

- [8] 5. The solution of the wave equation $u_{xx} = \frac{1}{c^2} u_{tt}$, $0 < x < L$, which satisfies the boundary conditions $u(0, t) = u(L, t) = 0$, has the form

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right].$$

Find the solution of $u_{xx} = u_{tt}$, $0 < x < 1$, which satisfies the boundary conditions $u(0, t) = u(1, t) = 0$, and the initial conditions $u(x, 0) = 0$, $u_t(x, 0) = x - x^2$. Write down the

complete solution $u(x, t)$.

Solution:

Here, $c = 1$ and $L = 1$, and $u(x, t) = \sum_{n=1}^{\infty} \sin(n\pi x)[a_n \cos(n\pi t) + b_n \sin(n\pi t)]$.

Since $u(x, 0) = 0$, $a_n = 0$, $n \geq 1$.

$$x - x^2 = u_t(x, 0) = \sum_{n=1}^{\infty} n\pi b_n \sin(n\pi x) \Rightarrow$$

$$\begin{aligned} n\pi b_n &= 2 \int_0^1 (x - x^2) \sin(n\pi x) dx \\ &= -\frac{2}{n\pi} (x - x^2) \cos(n\pi x) \Big|_0^1 + \frac{2}{n\pi} \int_0^1 (1 - 2x) \cos(n\pi x) dx \\ &= \frac{2}{n^2\pi^2} (1 - 2x) \sin(n\pi x) \Big|_0^1 + \frac{4}{n^2\pi^2} \int_0^1 \sin(n\pi x) dx \\ &= -\frac{4}{n^3\pi^3} \cos(n\pi x) \Big|_0^1 = \frac{4}{n^3\pi^3} [1 - (-1)^n]. \end{aligned}$$

$$\text{Hence, } u(x, t) = \sum_{n=1}^{\infty} \frac{4[1 - (-1)^n]}{n^3\pi^3} \sin(n\pi x) \sin(n\pi t).$$