Problem 1 [Sec. 1.2: # 12]

I will show that the sum of two even functions is even and the scalar multiplication too. So, if f, g are even, then

$$(f+g)(t) \stackrel{\text{def}}{=} f(t) + g(t) \stackrel{\text{symmetry}}{=} f(-t) + g(-t) \stackrel{\text{def}}{=} (f+g)(-t) \tag{1}$$

So the sum of even functions is even. Clearly the null function (which I denote by $\mathbf{0}$) is also even because $\mathbf{0}(t) = 0 = \mathbf{0}(-t)$. Finally if $c \in \mathbb{R}$ is a scalar, then $(cf)(t) = c\,f(t) = c\,f(-t) = (cf)(-t)$ is also even. Therefore the set of even functions is closed under the already defined operations of addition and mult. by scalar in Ex. 3. The remainder of the axioms is verified exactly as in the exercise.

Problem 2 [Sec. 1.2: # 18] We need to check all the axioms of vector space and see if any of those fails. The very first axiom (VS1) (commutativity of vector addition) fails.

Note that to *disprove* an axiom (or any mathematical statement) it suffices to display a **counterexample** (which may be very specific, even using numbers).

For example, let

$$(0,0) \boldsymbol{+} (1,1) = (0 + 2 \cdot 1, 0 + 3 \cdot 1) = (2,3) \tag{2}$$

however

$$(1,1) \boldsymbol{+} (0,0) = (1 + 2 \cdot 0, 1 + 3 \cdot 0) = (1,1) \tag{3}$$

and thus (VS1) is not verified. Note that the symbol $+$ is used in two different senses on the left side and in the components on the right side, which is why, if you look carefully, the $+$ on the left side is a boldface symbol.

Problem 3 [Sec. 1.3: # 19] We need to prove the statement in both directions; we start with the sufficiency.
Hypothesis W_1, W_2 are subspaces and $W_1 \subseteq W_2$ (or $W_1 \subseteq W_2$)
Thesis $W_1 \cup W_2$ is a vector subspace.
We prove it in the case $W_1 \subseteq W_2$ (because the other option is proven similarly). In this case $W_1 \cup W_2 = W_2$ and since W_2 is a subspace by assumption, then $W_1 \cup W_2$ is a subspace too.

Necessity: **Hypothesis**: W_1, W_2 and $W_1 \cup W_2$ are vector subspaces.
Thesis: $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. By contradiction, suppose that W_1 is not contained in W_2 **and** W_1 is not contained in W_2. Then there are two vectors $a \in W_1$ (but $a \notin W_2$) and $b \in W_2$ but $b \notin W_1$. Then I claim that $a + b$ is neither in W_1 nor in W_2 and thus not in $W_1 \cup W_2$; to show the claim, let us call $c = a + b$. Then, if $c \in W_1$ then $b = a - c \in W_1$ (not possible by our choice). Similarly, if $c \in W_2$ then $a = c - b \in W_2$ (again contradiction).

It may be useful to support this proof with an example (which is *not* a proof by itself, and it is only provided for convenience). In $\mathbb{R}^2$ conside the subspaces $W_1 = \{(x, 0),\ x \in \mathbb{R}\}$ and $W_2 = \{(0, y),\ y \in \mathbb{R}\}$. Then $W_1 \cup W_2$ is the union of the two axes. However the vector $(1,0) + (0,2) = (1,2)$ does not belong to any of the axes.

Problem 4 [Sec. 1.3: # 23] **(a)** We need to show that $W_1 + W_2$ is a subspace. Using the Criterion Thm 1.3.

1. since $0 \in W_1$, and $0 \in W_2$ then $0 + 0 = 0$ belongs to $W_1 + W_2$;
2. Let $z, z' \in W_1 + W_2$ we need to show $z + z' \in W_1 + W_2$. Necessarily (by definition of space sum given above the exercise) there are vectors $x, x' \in W_1$ and $y, y' \in W_2$ such that
$$z = x + y \ , \qquad z' = x' + y'. \tag{4}$$
But then
$$z + z' = (x + y) + (x' + y') \stackrel{VS1\&VS2}{=} \underbrace{(x + x')}_{\in W_1} + \underbrace{(x + y)}_{\in W_2} \tag{5}$$
and thus $z + z' \in W_1 + W_2$.
3. if $c \in \mathbb{R}$ and z is as before, then
$$cz = c(x + y) \stackrel{VS7}{=} \underbrace{cx}_{\in W_1} + \underbrace{cy}_{\in W_2} \tag{6}$$
and so $cz \in W_1 + W_2$.

As for the statement that $W_1 + W_2$ contains both W_1, W_2 this follows because all vectors $x \in W_1$ and $y \in W_2$ can be written as $x + 0$ and $0 + y$ respectively, where the null vector 0 is thought of as belonging to W_2 or W_1 respectively.

(b) Let X be a subspace of V containing both W_1, W_2. Let $z \in W_1 + W_2$ be written as $z = x + y$ with $x \in W_1, y \in W_2$. Since x, y must belong to X (by assumption $X \supseteq W_j$) **and** X is a subspace, then $z \in X$. Thus X contains all elements of $W_1 + W_2$.

Problem 5 [Sec. 1.3: # 24] We need to show two things:

1. $F^n = W_1 + W_2$;
2. $W_1 \cap W_2 = \{0\}$.

As for the first, clearly
$$(a_1, a_2, \ldots, a_{n-1}, a_n) = \underbrace{(a_1, a_2, \ldots, a_{n-1}, 0)}_{\in W_1} + \underbrace{(\overbrace{0, 0, \ldots, 0}^{n-1 \text{ times}}, a_n)}_{\in W_2} \tag{7}$$

The second point is also easily seen; a vector belongs to W_1 iff the last entry is zero, and it belongs to W_2 iff the first $n - 1$ entries are zero, so it belongs to the intersection iff all the entries are zero (i.e. it is the null vector).