

# LINEAR ALGEBRA WITH APPLICATIONS TO BUSINESS AND ECONOMICS

MATH1119B

LECTURE 2-2

INVERSE OF A MATRIX

# INVERSE OF A MATRIX

- An  $n \times n$  matrix  $A$  is said to be invertible if there is an  $n \times n$  matrix  $C$  such that

$$CA = I \text{ and } AC = I$$

- In this case,  $C$  is an inverse of  $A$
- $C$  is uniquely determined by  $A$ , because if  $B$  were another inverse of  $A$ , then

$$B = BI = B(AC) = (BA)C = IC = C$$

- This unique inverse is denoted by  $A^{-1}$  such that

$$A^{-1}A = I \text{ and } AA^{-1} = I$$

# INVERSE OF A MATRIX

- A matrix that is invertible is called a non-singular matrix.
- A matrix that is not invertible is called singular matrix

Theorem 4

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . If  $ad - bc \neq 0$ , then  $A$  is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If  $ad - bc = 0$ , then  $A$  is not invertible

Where  $ad - bc$  is called the determinant of  $A$ .

# INVERSE OF A MATRIX

- $\det A = ad - bc$
- By theorem 4, 2x2 matrix A is invertible if and only if  $\det A \neq 0$

Ex2-2-1 Find the inverse of the following matrices

a.  $\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$     b.  $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$

# INVERSE OF A MATRIX

## Theorem 5

If  $A$  is an invertible  $n \times n$  matrix, then for each  $b$  in  $\mathcal{R}^n$ , the equation  $Ax = b$  has the unique solution  $x = A^{-1}b$

# INVERSE OF A MATRIX

## Theorem 6

- a. If  $A$  is invertible, then  $A^{-1}$  is invertible and  $(A^{-1})^{-1} = A$
  - b. If  $A$  and  $B$  are  $n \times n$  invertible matrices, then so is  $AB$ , and the inverse of  $AB$  is the product of the inverse of  $A$  and  $B$  in the reverse order. That is  $(AB)^{-1} = B^{-1}A^{-1}$
  - c. If  $A$  is invertible matrix, then so is  $A^T$ , and the inverse of  $A^T$  is the transpose of  $A^{-1}$ . That is  $(A^T)^{-1} = (A^{-1})^T$
- An invertible matrix  $A$  is row equivalent to an Identity matrix and we can find  $A^{-1}$  by watching the row reduction of  $A$  to  $I$

# ELEMENTARY MATRICES

- An elementary matrix is one that is obtained by a single elementary row operation on an identity matrix.

Ex 2-2-2

Calculate  $E_1A$ ,  $E_2A$ ,  $E_3A$  when

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

# ELEMENTARY MATRICES

- If an elementary row operation is performed on an  $m \times n$  matrix  $A$ , the resulting matrix can be written as  $EA$ , where the  $m \times m$  matrix  $E$  is created by performing the same row operation on  $I_m$
- We know that row operations are reversible. If  $E$  is produced by row operations on  $I$ , then there is another row operation of the same type that changes  $E$  back to  $I$ . Therefore, there is an elementary matrix  $F$  such that  $FE = I$
- Each elementary matrix  $E$  is invertible. The inverse of  $E$  is the elementary matrix of the same type that transforms  $E$  back to  $I$ .

# ELEMENTARY MATRICES

Ex 2-2-3

Find the inverse of the following matrices

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

# INVERSE OF A MATRIX

## Theorem 7

An  $n \times n$  matrix  $A$  is invertible if and only if  $A$  is row equivalent to  $I_n$ , and in this case, any sequence of elementary row operations that reduces  $A$  to  $I_n$  also transforms  $I_n$  into  $A^{-1}$ .

- Algorithm for finding  $A^{-1}$ : Row reduce the augmented matrix  $[A \ I]$ . If  $A$  is row equivalent to  $I$ , then  $[A \ I]$  is row equivalent to  $[I \ A^{-1}]$ . Otherwise,  $A$  does not have an inverse.

# INVERSE OF A MATRIX

Ex2-2-4

Find the inverse of the following matrices.

$$A = \begin{bmatrix} 5 & 10 \\ 4 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

# ANOTHER VIEW OF MATRIX INVERSION

- Let us denote the columns of  $I_n$  by  $e_1, e_2, \dots, e_n$ .
- The row reduction of  $[A \ I]$  to  $[I \ A^{-1}]$  can be viewed as the simultaneous solution of the  $n$  systems

$$Ax = e_1, Ax = e_2, \dots, Ax = e_n$$

Where the “augmented columns” of these systems have all been placed next to  $A$  to form

$$[A \ e_1 \ e_2 \ \dots \ e_n] = [A \ I]$$

- The equation  $AA^{-1} = I$  and the definition of matrix multiplication show that the columns of  $A^{-1}$  are the solutions of the above  $n$  systems