# Linear Algebra with Applications to Business and Economics

MATH1119B

Chapter 2-1

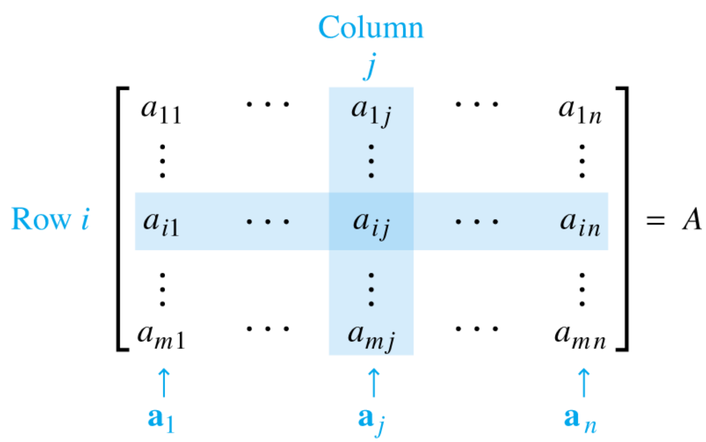
Matrix Operations

# MATRIX NOTATIONS

- If A is an mxn matrix (a matrix with m rows and n columns), then the scaler entry in the $i^{th}$ row and $j^{th}$ column of A is denoted by $a_{ij}$ and called the (i, j) entry of A.

$$A = [\boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \cdots \boldsymbol{a_{n-1}} \quad \boldsymbol{a_n}] = [a_{ij}]_{mxn}$$

Where $\boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \cdots \boldsymbol{a_{n-1}} \quad \boldsymbol{a_n}$ are vectors with m real numbers i.e. vectors in $\mathscr{R}^m$




Matrix notation.

# MATRIX NOTATIONS

- The number $a_{ij}$ is the $i^{th}$ entry of the column vector $\mathbf{a}_j$
- The diagonal entries in an mxn matrix $A = [a_{ij}]$ are $a_{11}, a_{22}, a_{33}, \cdots$ and they form the main diagonal of A.
- A diagonal matrix is a square matrix (number of columns = number of rows) whose non-diagonal entries are zero.
- An example is the nxn identity matrix, $I_n$.
- An mxn matrix whose entries are all zero is a zero matrix and is written as 0

# MATRIX OPERATIONS

- Two matrices are equal if they have the same dimension (i.e. number of columns and number of rows of both matrices are equal) and their corresponding entries are equal
- If A and B are mxn matrices, then the sum A+B is the mxn matrix whose columns are the sums of the corresponding columns in A and B.
- If r is a scaler and A is a matrix, then the scalar multiple rA is the matrix whose columns are r times the corresponding columns in A.

# Matrix Operations

**Theorem 1**: Let A, B, and C be matrices of the same size, and let r and s be scalars

a) $A+B = B+A$
b) $(A+B) +C = A+(B+C)$
c) $r(A+B) = rA + rB$
d) $(r+s)A = rA+sA$
e) $r(sA) = (rs)A$

# EXERCISE

Ex2.1.1

Compute A+B and A+C. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix};\ B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix};\ C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

Ex2.1.2

Compute 2B and A-2B for matrices A and B in Ex2.1.1

# MATRIX MULTIPLICATION

- Let A be an mxn matrix and B an rxs matrix
- The multiplication of the matrix A by B, written AB is defined if and only if n=r. (i.e. number of column of A must be equal to the number of rows of B)
- If n = r, then AB equal a new matrix C whose dimension will be mxs

# MATRIX MULTIPLICATION

- If A is an mxn matrix, and if B is nxp matrix with columns $b_1, b_2, \cdots, b_p$,then the product AB is the mxp matrix whose columns are $Ab_1, Ab_2, \cdots, Ab_p$. i.e

$$AB = A\left[b_1\ b_2 \cdots b_p\right] = \left[Ab_1\ Ab_2 \cdots Ab_p\right]$$

# EXERCISE

Ex2.1.3

Compute CB

$$C = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}$$

# MATRIX MULTIPLICATION

- If AB exists, then $AB = C$, where $C = [c_{ij}]_{mxn}$, where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

I.e., the (i, j)$^{th}$-entry of C consists of the sum of the product of each entry on the i$^{th}$ row by the corresponding entries of the j$^{th}$ columns of the matrix B.

- The definition of AB shows that AB has the same number of rows as A and the same number of columns as B
- In general for matrices A  and, B such that AB and BA exists, we do not have $AB = BA$

# EXERCISE

Ex2.1.4

Compute A-5$I_3$ and 5$I_3$A when

$$A = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}$$

Ex2.1.5

Compute AC and CA

$$C = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}; \quad A = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$$

# PROPERTIES OF MATRIX MULTIPLICATION

**Theorem 2**

Let A be an mxn matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. A(BC) = (AB)C (associative law of multiplication)

b. A(B+C) = AB+AC (left distributive law)

c. (B+C)A = BA + CA (right distributive law)

d. r(AB) =(rA)B = A(rB) for any scaler r

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

# WARNINGS

- In general, $AB \neq BA$
- The cancellation laws do not hold for matrix multiplication. i.e. if AB = AC, then it is not true in general that B = C
- If the product AB is a the zero matrix, then you cannot conclude in general that either A = 0 or B = 0

# EXERCISE

Ex2.1.6

a. Compute AC and CA when $C = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$; $A = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$

b. Compute AB and AC when $A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$;
$C = \begin{bmatrix} 2 & 7 \\ 3 & 4 \end{bmatrix}$

c. Compute AB when $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

# Powers of Matrix

- If A is an nxn matrix and if k is a positive integer, then $A^k$ denotes the product of k number of matrix of A

i.e.

$$A^k = \underbrace{A \cdots A}_{k}$$

# THE TRANSPOSE OF A MATRIX

- Given an mxn matrix A, the transpose of A is the nxm matrix, denoted by $A^T$, whose columns are formed from the corresponding rows of A.

# EXERCISE

Ex2.1.7

Compute the following:

a. $A^2$ , $A^T$ when $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

b. $A^2$ when $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$

c. $A^T$ when $A = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}$

d. $I^2$ and $I^T$ when $I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

# TRANSPOSE OF A MATRIX

**Theorem 3**

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) For any scaler r, $(rA)^T = rA^T$

d) $(AB)^T = B^T A^T$ (i.e. The transpose of a product of matrices equals the product of their transposes in the *reverse* order.)

# EXERCISE

Ex2.1.8

Compute the following when r= 2

$A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$

a. $(A^T)^T$

b. $(A+B)^T$ and $A^T + B^T$

c. $(rA)^T$ and $rA^T$

d. $(AB)^T$ and $B^TA^T$