Name ____________________ Student #___________________
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1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 1 \\ 7 & 8 & 9 \end{bmatrix}$.

a) Find A^{-1}, if it exists.

ANS:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 1 \\ 7 & 8 & 9 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 6 & 5 & 1 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \sim \begin{pmatrix} R_2' = R_2 - 6R_1 \\ R_3' = R_3 - 7R_1 \end{pmatrix} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -7 & -17 & -6 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right]$$

$$\sim (R_3' = R_3/(-6)) \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -7 & -17 & -6 & 1 & 0 \\ 0 & 1 & 2 & 7/6 & 0 & -1/6 \end{array}\right] \sim (R_3 \leftrightarrow R_2) \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 7/6 & 0 & -1/6 \\ 0 & -7 & -17 & -6 & 1 & 0 \end{array}\right]$$

$$\sim (R_3' = R_3 + 7R_2) \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 7/6 & 0 & -1/6 \\ 0 & 0 & -3 & 13/6 & 1 & -7/6 \end{array}\right] \sim (R_3' = R_3/(-3)) \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 7/6 & 0 & -1/6 \\ 0 & 0 & 1 & -13/18 & -1/3 & 7/18 \end{array}\right]$$

$$\sim \begin{pmatrix} R_2' = R_2 - 2R_3 \\ R_1' = R_1 - 3R_3 \end{pmatrix} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 57/18 & 1 & -21/18 \\ 0 & 1 & 0 & 47/18 & 2/3 & -17/18 \\ 0 & 0 & 1 & -13/18 & -1/3 & 7/18 \end{array}\right]$$

$$\sim (R_1' = R_1 - 2R_2) \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -37/18 & -1/3 & 13/18 \\ 0 & 1 & 0 & 47/18 & 2/3 & -17/18 \\ 0 & 0 & 1 & -13/18 & -1/3 & 7/18 \end{array}\right]$$

b) Evaluate $\det A$ by doing the cofactor expansion along the second column of A.

ANS:

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 6 & 1 \\ 7 & 9 \end{vmatrix} = -(6.9 - 1.7) = -47$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = (1.9 - 3.7) = -12$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 6 & 1 \end{vmatrix} = -(1.1 - 3.6) = 17$$

$$\det A = 2.C_{12} + 5.C_{22} + 8.C_{32} = 2.(-47) + 5.(-12) + 8.17 = -18$$

2. Evaluate $\det A$ by reducing A using elementary row/column operations.

$$A = \begin{bmatrix} 1 & 0 & 1 & -2 & 3 \\ 2 & 2 & 2 & -3 & 4 \\ 3 & 3 & 6 & -6 & 9 \\ 4 & 4 & 8 & -8 & 5 \\ 5 & 6 & 10 & -9 & 10 \end{bmatrix}$$

ANS:

$$\det A=\begin{vmatrix}1&0&1&-2&3\\2&2&2&-3&4\\3&3&6&-6&9\\4&4&8&-8&5\\5&6&10&-9&10\end{vmatrix}(C_3'=C_3+C_4)=\begin{vmatrix}1&0&-1&-2&3\\2&2&-1&-3&4\\3&3&0&-6&9\\4&4&0&-8&5\\5&6&1&-9&10\end{vmatrix}(R_1'=R_1-R_2)$$

$$=\begin{vmatrix}-1&-2&0&1&-1\\2&2&-1&-3&4\\3&3&0&-6&9\\4&4&0&-8&5\\5&6&1&-9&10\end{vmatrix}(R_2'=R_2+R_5)=\begin{vmatrix}-1&-2&0&1&-1\\7&8&0&-12&14\\3&3&0&-6&9\\4&4&0&-8&5\\5&6&1&-9&10\end{vmatrix}$$

$$=(-1)^{5+3}.1\begin{vmatrix}-1&-2&1&-1\\7&8&-12&14\\3&3&-6&9\\4&4&-8&5\end{vmatrix}(C_1'=C_1-C_2)$$

$$=\begin{vmatrix}1&-2&1&-1\\-1&8&-12&14\\0&3&-6&9\\0&4&-8&5\end{vmatrix}(R_1'=R_1+R_2)=\begin{vmatrix}0&6&-11&-1\\-1&8&-12&14\\0&3&-6&9\\0&4&-8&5\end{vmatrix}$$

$$=(-1)^{2+1}(-1)\begin{vmatrix}6&-11&-1\\3&-6&9\\4&-8&5\end{vmatrix}(C_2'=C_2+2C_1)=\begin{vmatrix}6&1&-1\\3&0&9\\4&0&5\end{vmatrix}$$

$$=(-1)^{1+2}1\begin{vmatrix}3&9\\4&5\end{vmatrix}=-(3.5-4.9)=21$$

3. Use Cramer's rule to solve

$$\begin{aligned}2x-y+3z&=-3\\-x-y+3z&=-6\\x-2y-z&=-2\end{aligned}$$

ANS:

$$A=\begin{bmatrix}2&-1&3\\-1&-1&3\\1&-2&-1\end{bmatrix},\ b=\begin{bmatrix}-3\\-6\\-2\end{bmatrix}$$

$$\det A=\begin{vmatrix}2&-1&3\\-1&-1&3\\1&-2&-1\end{vmatrix}\left(R_1'=R_1-R_2\right)=\begin{vmatrix}3&0&0\\-1&-1&3\\1&-2&-1\end{vmatrix}=(-1)^{1+1}3\begin{vmatrix}-1&3\\-2&-1\end{vmatrix}=3.(1+6)=21$$

$$\det A_1(b)=\begin{vmatrix}-3&-1&3\\-6&-1&3\\-2&-2&-1\end{vmatrix}\left(R_1'=R_1-R_2\right)=\begin{vmatrix}3&0&0\\-6&-1&3\\-2&-2&-1\end{vmatrix}=(-1)^{1+1}3\begin{vmatrix}-1&3\\-2&-1\end{vmatrix}=3.(1+6)=21$$

$$\det A_2(b)=\begin{vmatrix}2&-3&3\\-1&-6&3\\1&-2&-1\end{vmatrix}\left(R_1'=R_1-R_2\right)=\begin{vmatrix}3&3&0\\-1&-6&3\\1&-2&-1\end{vmatrix}=(-1)^{1+1}3\begin{vmatrix}-6&3\\-2&-1\end{vmatrix}+(-1)^{1+2}3\begin{vmatrix}-1&3\\1&-1\end{vmatrix}=3.12-3.(-2$$

$$\det A = \begin{vmatrix} 2 & -1 & -3 \\ -1 & -1 & -6 \\ 1 & -2 & -2 \end{vmatrix} \left(R_1' = R_1 - R_2\right) = \begin{vmatrix} 3 & 0 & 3 \\ -1 & -1 & -6 \\ 1 & -2 & -2 \end{vmatrix} = (-1)^{1+1} 3 \begin{vmatrix} -1 & -6 \\ -2 & -2 \end{vmatrix} + (-1)^{1+3} 3 \begin{vmatrix} -1 & -1 \\ 1 & -2 \end{vmatrix} = 3.(2-12) + 3.3$$

So.

$$x = \frac{\det A_1(b)}{\det A} = \frac{21}{21} = 1,\ y = \frac{\det A_2(b)}{\det A} = \frac{42}{21} = 2,\ z = \frac{\det A_2(b)}{\det A} = \frac{-21}{21} = -1$$