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1. Given the matrix A and an echelon form R, find bases for each of the column space of A, row space of A and the null space of A.

$$A=\begin{bmatrix}2 & -4 & 0 & 2 & 1\\ -1 & 2 & 1 & 2 & 3\\ 1 & -2 & 1 & 4 & 4\end{bmatrix}\sim\begin{bmatrix}1 & -2 & 1 & 4 & 4\\ 0 & 0 & 2 & 6 & 7\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}=R$$

ANS: The columns of R that have pivots are columns 1 and 3.
So, column 1 and column 3 of A form the basis for column space of A.

- for row space of A: from R (reduced)
- for column space of A: from A (original)

i.e. $\left\{\begin{bmatrix}2\\-1\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ form the basis for column space of A.

The rows that have pivots are rows 1 and 2.
So, row 1 and row2 of R form the basis for row space of A.
i.e. $\{[1\ \ -2\ \ 1\ \ 4\ \ 4],[0\ \ 0\ \ 2\ \ 6\ \ 7]\}$ form the basis for row space of A.

Since 3 columns column (2, 4 and 5) in R have no pivot, the solution of $AX=0$ has 3 free variables.
Let $x_5=t, x_4=s, x_2=r$.

Then $x_3=\dfrac{-6s-7t}{2}=-3s-\dfrac{7t}{2}$. and $x_1=2r-(-3s-\dfrac{7t}{2})-4s-4t=2r-s-\dfrac{t}{2}$

$$AX=0\Rightarrow X=\begin{bmatrix}2r-s-\frac{t}{2}\\ r\\ -3s-\frac{7t}{2}\\ s\\ t\end{bmatrix}=\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}r+\begin{bmatrix}-1\\0\\-3\\1\\0\end{bmatrix}s+\begin{bmatrix}-\frac{1}{2}\\0\\-\frac{7}{2}\\0\\1\end{bmatrix}t$$

Then

$\left\{\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\-3\\1\\0\end{bmatrix},\begin{bmatrix}-\frac{1}{2}\\0\\-\frac{7}{2}\\0\\1\end{bmatrix}\right\}$ form the basis for the null space of A.

2. Determine if the columns of the matrix form a basis for $\Re^4$.
Justify your answer.

$$\begin{bmatrix} -4 & -3 & 0 & 1 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix}$$

- Span: whether system is consistent
- whether it is linearly independent set ($\det \neq 0$).

ANS:

$$\begin{bmatrix} -4 & -3 & 0 & 1 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ -4 & -3 & 0 & 1 \\ 5 & 4 & 6 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & -3 & 12 & 1 \\ 0 & 4 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 7 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since all 4 columns have pivot positions, the columns of the matrix form a linearly independent set. Since all 4 rows have pivot positions, the rows of the matrix form a basis for $\Re^4$. Therefore columns of the matrix form a basis for $\Re^4$.

3. Let $A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$

where R is an echelon form of the matrix A.

a) Find a basis for $ColA$ and a basis for $NulA$.

ANS:

The columns of R that have pivots are columns 1,2 and 5.

So, column 1,column 2 and column 5 of A form the basis for column space of A.

i.e. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \\ 5 \end{bmatrix} \right\}$ form the basis for column space of A.

Since 2 columns column (3, 4) in R have no pivot, the solution of $AX = 0$ has 2 free variables.

Let $x_4 = t, x_3 = s$.

Then $x_5 = 0, x_2 = -s - 3t$. and $x_1 = 2(-s - 3t) - s - t = -3s - 7t$

$$AX = 0 \Rightarrow X = \begin{bmatrix} -3s - 7t \\ -s - 3t \\ s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} t$$

Then

$$\left\{\begin{bmatrix}-3\\-1\\1\\0\\0\end{bmatrix},\begin{bmatrix}-7\\-3\\0\\1\\0\end{bmatrix}\right\}$$ form the basis for the null space of A.

b) Find $\dim ColA$ and $\dim NulA$. Verify the rank theorem.

ANS:

$\dim ColA$ =# of vectors in $ColA$=3=rank A

$\dim NulA$ =# of vectors in $NulA$=2=Nullity A

According to Rank Theorem,

Rank A +Nullity A=# of columns in A which holds as 2+3=5= # of columns of A.

c) Determine if the vector $\begin{bmatrix}-2\\3\\-3\\-1\end{bmatrix}$ is in the column space of A. Explain.

ANS: we need to check if

$$\begin{bmatrix}1&-2&2\\-1&3&-2\\0&1&4\\1&2&5\end{bmatrix}X=\begin{bmatrix}-2\\3\\-3\\-1\end{bmatrix}$$ is consistent system (any solution exists) or not.

$$\left[\begin{array}{ccc|c}1&-2&2&-2\\-1&3&-2&3\\0&1&4&-3\\1&2&5&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&0&2\\0&1&0&1\\0&0&1&-1\\0&0&0&0\end{array}\right]$$ is consistent system. So the given vector is in the column space of A.

d) Determine if the vector $\begin{bmatrix}4\\2\\-1\\1\\0\end{bmatrix}$ is in the null space of A. Explain.

ANS: We need to check if the given vector is a solution to $AX=0$.

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.4-2.2+1.(-1)+1.1+2.0 \\ (-1).4+3.2+0.(-1)+2.1+(-2).0 \\ 0.4+1.2+1.(-1)+3.1+4.0 \\ 1.4+2.2+5.(-1)+13.1+5.0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore,

the vector $\begin{bmatrix} 4 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ is **not** in the null space of A.

4. Let $v_1 = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -11 \\ 6 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 2 \\ -9 \\ h \end{bmatrix}$. For what values of h, do $\{v_1, v_2, v_3\}$ form a basis for $\Re^3$?

ANS: We need to check if $\{v_1, v_2, v_3\}$ form a linearly independent set and span $\Re^3$.

$$\begin{bmatrix} 1 & 2 & 2 \\ -5 & -11 & -9 \\ -3 & 6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 12 & h+6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & h+18 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & h+18 \end{bmatrix}$$

If $h+18=0$, then $\{v_1, v_2, v_3\}$ will neither be linearly independent nor for can span $\Re^3$.
Therefore for all $h \neq -18$, $\{v_1, v_2, v_3\}$ form a basis for $\Re^3$.

$\det A = -11h + 54 - 60 - 66 + 54 + 10h \neq 0$

$\Rightarrow -h - 18 \neq 0 \Rightarrow h \neq -18$