A quick introduction

The focus of this course is on an area of mathematics called *Linear Algebra*

The basic idea of linear algebra is to solve *systems of linear equations*.

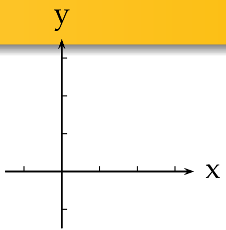
$$\text{systems of linear equations} \quad \rightsquigarrow \quad \text{matrices}$$

Matrices appear in:

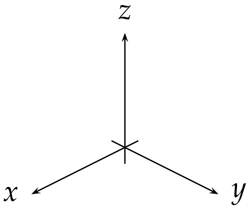
- **Mathematical modelling of economy (Leontief model,...)**
- **eigenvalues and eigenvectors (Google)**
- **least squares method (GPS systems)**
- **Markov chains (Population studies, stock market)**
- ...

$\mathbb{R}^2$ and $\mathbb{R}^3$

$\mathbb{R}^2$ is just the usual xy-plane.



$\mathbb{R}^3$ is just the usual xyz-space.



Our goal in linear algebra is to study questions related to linear objects in $\mathbb{R}^2, \mathbb{R}^3, \ldots$

by linear objects we mean lines, planes, etc.

A **linear equation** in the variables $x_1, \cdots, x_n$ is an equation of the form

$$a_1x_1 + \cdots + a_nx_n = b$$

where $a_1, ..., a_n, b$ are constants.

$a_1, ..., a_n$ are called the **coefficients** of the linear equation.

Main idea: in a linear equation we don't want x_ix_j or x_i^2 or x_i^3 or ....

Example

$$3x_1 - 2x_2 + x_3 = -3$$

is a linear equation.

$$\sqrt{2}x_1 - 1 + \frac{2}{\sqrt{3}}x_2 = x_3$$

is a linear equation because we can write it in the form

$$\sqrt{2}x_1 + \frac{2}{\sqrt{3}}x_2 - x_3 = 1$$

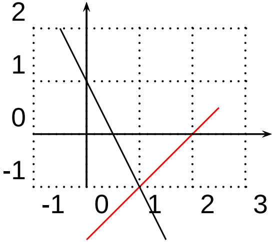
$$x_1x_2 - x_1 + x_3 = 1$$

Solving systems of linear equations

Consider the following equation: $y + 2x = 1$.

It is the equation of a line and it can be sketched as follows. Every point (x, y) on the line is a *solution* to the linear equation.

If we also sketch the line $y - x = -2$ then we obtain the following picture:

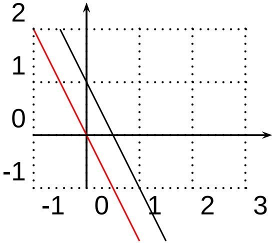


The solution to the linear system $\begin{cases} y + 2x = 1 \\ y - x = -2 \end{cases}$ is the point of intersection of the two lines, i.e. $(1, -1)$.

Solving systems of linear equations

Consider the following linear system: $\begin{cases} y + 2x = 1 \\ -x - \frac{1}{2}y = 0 \end{cases}$.

To solve the system we can draw the two lines to find their intersection point. But this time the lines are parallel.



Therefore the linear system does *not* have a solution. We call such a linear system inconsistent.

Linear Systems

Recall that a system of linear equations is a collection of linear equations in the same set of variables.

Example

$$\begin{cases} 2x_1 & -3x_2 & +x_3 & +x_4 & = 3 \\ x_1 & -x_2 & -x_3 & & = 2 \\ & x_2 & -x_3 & +x_4 & = 3 \end{cases}$$

The solution set of a linear system is the set of all lists of numbers $(s_1, \ldots, s_n)$ such that the values $x_1 = s_1, \ldots\ x_n = s_n$ solve the system.

Solution Set

Example

$$\begin{cases} 2x_1 & -3x_2 & +x_3 & +x_4 & = 3 \\ x_1 & -x_2 & -x_3 & & = 2 \\ & x_2 & -x_3 & +x_4 & = 3 \end{cases}$$

- $x_1 = 1, x_2 = 0, x_3 = -1, x_4 = 2$ is a solution of this system.

$$\begin{cases} 2(1) & -3(0) & +(-1) & +2 & = 3 \\ 1 & -(0) & -(-1) & & = 2 \\ & 0 & -(-1) & +2 & = 3 \end{cases}$$

Question. Are there other solutions?

Gaussian elimination is one of the methods for solving a linear equation.

Idea: Each time we eliminate one of the variables by adding/subtracting suitable multiples of one equation to/from the other.

Example. Solve
$$\begin{cases} 2u & +v & +w & = 5 \\ 4u & -6v & & = -2 \\ -2u & +7v & +2w & = 9 \end{cases}$$

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First we get rid of u in the last two equations. To do so, we

(a) subtract 2 times the first equation from the second
(b) Add the first equation to the third.
We obtain the system

$$\begin{cases} 2u & +v & +w & = 5 \\ (4-2(2))u & +(-6-2(1))v & +(0-2)w & = -2-2(5) \\ (-2+2)u & +(7+1)v & +(2+1)w & = 9+5 \end{cases}$$

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We now get rid of v. To do so, we

(c) Add the second equation to the third.

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In the last slide we obtained the following system:

$$\begin{cases} 2u & +v & +w & = 5 \\ & -8v & -2w & = -12 \\ & & w & = 2 \end{cases}$$

The system is now solved backwards.

$w = 2$

$-8v = 2w - 12 = 4 - 12 = -8 \Longrightarrow v = 1$

$2u = 5 - v - w = 5 - 1 - 2 = 2 \Longrightarrow u = 1.$

unique solution

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unique solution

Example. Solve the system $\begin{cases} x_1 & +x_2 & -x_3 & = 1 \\ x_1 & & +x_3 & = -1 \\ & -x_2 & +2x_3 & = 0 \end{cases}$

Again we try to eliminate variables. To eliminate x_1 :

• subtract the first equation from the second equation.

But then we get $\begin{cases} x_1 & +x_2 & -x_3 & = 1 \\ & -x_2 & +2x_3 & = -2 \\ & -x_2 & +2x_3 & = 0 \end{cases}$

Should we really go on?!

No, because the second and the third equation of the new system are inconsistent!

This system does NOT have a solution. It is called inconsistent.

Example. Solve the following system. $\begin{cases} x & -2y & = 1 \\ 2x & -4y & = 2 \end{cases}$

To solve this system note that the two equations are essentially the same :

$x - 2y = 1 \Rightarrow x = 2y + 1.$

$2x - 4y = 2 \Rightarrow 2x = 4y + 2 \Rightarrow x = 2y + 1.$

Any two numbers x, y such that $x = 2y + 1$ solve the system.

$x = 3, y = 1$
$x = -1, y = -1$
...

There are infinitely many solutions.

In general, a linear system of equations can have

- No solutions (it is called inconsistent.)
- A unique solution (it is called consistent.)
- Infinitely many solutions (it is still called consistent.)

Note that this is special to *linear* systems. For example, the (non-linear) equation $x^2 - 1 = 0$ has exactly two solutions: $x = \pm 1$.

In conclusion:

- If the linear system has a solution, we call it consistent.
- If the linear system does not have a solution, then we call it inconsistent.

Matrix notation

It would be very helpful to put the coefficients and constants of a linear system in an array (called a matrix).

There are two ways to do it. Consider the system

$$\begin{cases} x_1 & +x_2 & -x_3 & = 1 \\ x_1 & & +x_3 & = -1 \\ & -x_2 & +2x_3 & = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

augmented matrix

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

coefficient matrix

They simplify the elimination method.

Example

Solve $\begin{cases} 2u & +v & +w & = 5 \\ 4u & -6v & & = -2 \\ -2u & +7v & +2w & = 9 \end{cases}$ $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right]$

1. (a) subtract 2 times first row (R_1) from the second row (R_2).

   $R_2 \rightarrow R_2 - 2R_1$

   $\begin{cases} 2u & +v & +w & = 5 \\ & -8v & -2w & = -12 \\ -2u & +7v & +2w & = 9 \end{cases}$ $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ -2 & 7 & 2 & 9 \end{array}\right]$

2. (b) Add first row to the third row. $R_3 \rightarrow R_3 + R_1$

   $\begin{cases} 2u & +v & +w & = 5 \\ & -8v & -2w & = -12 \\ & +8v & +3w & = 14 \end{cases}$ $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right]$

3. Add second row to the third row $R_3 \rightarrow R_3 + R_2$

   $\begin{cases} 2u & +v & +w & = 5 \\ & -8v & -2w & = -12 \\ & & w & = 2 \end{cases}$ $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right]$ echelon form

Echelon Form

The idea is that the matrix entries have "staircase" structure.

The first nonzero entry of a row (from the left) is called a leading entry.

Example.

$$\begin{bmatrix} \mathbf{2} & -2 & 3 & 0 \\ 0 & \mathbf{-3} & 4 & 1 \\ 0 & 0 & 0 & \mathbf{-1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A matrix is in echelon form if

1. All nonzero rows are above all zero rows.
2. Each leading entry lies <u>to the right of</u> any leading entry <u>above</u> it.
3. All entries in a column below a leading entry are zeros.

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$$\begin{bmatrix} 2 & -2 & 3 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ is not in E.F. because R_3 is zero but R_4 is not zero.

$$\begin{bmatrix} 2 & -2 & 3 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is not in E.F. (look at the red entries).

$$\begin{bmatrix} -1 & -2 & 3 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is not in E.F. (look at the column of the red entry).

$$\begin{bmatrix} -1 & -2 & 3 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is in E.F.

Reduced Echelon Form

A matrix is in reduced echelon form (R.E.F.) if it is in E.F. and

1. All of the leading entries are equal to 1. (we call them leading 1's).
2. Each leading 1 is the only nonzero entry of its column.

Example. $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in E.F. but not in R.E.F.

$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in R.E.F.

Elementary row operations

Recall:

1. To solve a linear system, we use the elimination method.
2. The steps of the elimination process correspond to operations on the augmented matrix.

These operations are called the elementary row operations.

Elementary Row Operations

- Replace a row R_k with $R_k + cR_l$.
- Interchange two rows.
- Scale a row R_k by a scalar $c \neq 0$.

Example.

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -2 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0+2 & 1+(-2) & -2+0 & 0+4 \\ 2 & 0 & -1 & 1 \end{bmatrix} \quad R_2 \longrightarrow R_2 + 2R_1$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -2 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 0 & -1 & 1 \\ 0 & 1 & -2 & 0 \end{bmatrix} \quad R_2 \leftrightsquigarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -2 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -2 & 4 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix} \quad R_2 \longrightarrow -2R_2$$

Two matrices are called row equivalent if one can get from one to the other by a sequence of elementary row operations.

Important Fact

Given two linear systems, if their augmented matrices are row equivalent, then their solution set is the same.

This means that if we can solve one of them, then we have solved the other as well!

- To solve a linear system, use elementary row operations to put its augmented matrix into E.F. (or, even better, into R.E.F.)

Example.

Solve the system $\begin{cases} x_1 & +2x_2 & -x_3 & = 1 \\ & -x_2 & +x_3 & = 0 \\ x_1 & & +x_3 & = -1 \end{cases}$.

The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 \end{array}\right]$

We use elementary row operations.

1. $R_3 \to R_3 - R_1$ $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ \mathbf{0} & \mathbf{-2} & \mathbf{2} & \mathbf{-2} \end{array}\right]$
2. $R_3 \to R_3 - 2R_2$ $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-2} \end{array}\right]$ E.F.

But $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-2} \end{array}\right]$ is the augmented matrix of $\begin{cases} x_1 + 2x_2 - x_3 & = 1 \\ -x_2 + x_3 & = 0 \\ 0 & = -2 \end{cases}$

Therefore the system is **inconsistent**.

In general, if the E.F. (or R.E.F.) of the augmented matrix has a row of the form

$$\left[\begin{array}{ccccc|c} 0 & 0 & \cdots & 0 & 0 & a \end{array}\right]$$

where $a \neq 0$, then the original linear system is <u>inconsistent</u>.