

Total mark: 20. Closed book. No calculator is allowed.

Last Name _____ First Name _____ Student Number _____

Question 1. [4 Marks] Find $\frac{dy}{dx}$ given $x^2 - xy - y^2 - 2y = 1$

Solution: Differentiate both sides of the equation with respect to x , treating $y = f(x)$ as a differentiable function of x , then we have

$$2x - (y + xy') - 2yy' - 2y' = 0 \text{ [2 marks]}$$

Move all terms involving y' to the right side of the equation and move all other terms to the left side of the equation.

$$2x - y = (x + 2y + 2)y' \text{ [1 mark]}$$

then

$$y' = \frac{2x - y}{x + 2y + 2} \text{ [1 mark]}$$

Question 2. [3 Marks] Find the linearization $L(x)$ of $f(x) = \sqrt[3]{x}$ at $x = 1$ and use it to approximate $\sqrt[3]{0.997}$.

Solution: Since $f'(x) = \frac{1}{3}x^{-2/3}$, we have $f(1) = \sqrt[3]{1} = 1$ and $f'(1) = \frac{1}{3}x^{-2/3}|_{x=1} = \frac{1}{3}$. The linearization is

$$L(x) = 1 + \frac{1}{3}(x - 1)$$

and

$$\sqrt[3]{0.997} \approx 1 + \frac{1}{3}(0.997 - 1) = 1 - 0.001 = 0.999$$

Question 3. [9 Marks] Find the derivatives of the following functions:

(1) (3 points) $y = \ln(x^3 + x^2 - 1)$,

(2) (3 points) $y = \frac{2^x(x+1)^2}{(5x+1)^3}$

(3) (3 points) $y = [\ln(x)]^x$,

Solution:

1)

$$[\ln(x^3 + x^2 - 1)]' = \frac{1}{x^3 + x^2 - 1} [x^3 + x^2 - 1]' = \frac{3x^2 + 2x}{x^3 + x^2 - 1}$$

2) Apply \ln to both sides and use laws of logarithms:

$$\ln y = x \ln 2 + 2 \ln(x + 1) - 3 \ln(5x + 1) \text{ [1 mark]}$$

Take the derivative of both sides of this equality:

$$\begin{aligned} \frac{d}{dx} [\ln y] &= \frac{d}{dx} [x \ln 2 + 2 \ln(x + 1) - 3 \ln(5x + 1)] \\ \frac{y'}{y} &= \ln 2 + \frac{2}{x + 1} - \frac{15}{5x + 1} \text{ [1 mark]} \end{aligned}$$

Multiply both sides by y to get y'

$$\begin{aligned} y' &= y \left[\ln 2 + \frac{2}{x+1} - \frac{15}{5x+1} \right] [1 \text{ mark}] \\ &= \left[\frac{2^x(x+1)^2}{(5x+1)^3} \right] \left[\ln 2 + \frac{2}{x+1} - \frac{15}{5x+1} \right] \end{aligned}$$

3) Apply \ln to both sides and use laws of logarithms:

$$\ln y = x \ln(\ln x)$$

Take the derivative of both sides of this equality:

$$\begin{aligned} \frac{d}{dx} [\ln y] &= \frac{d}{dx} [x \ln(\ln x)] \\ \frac{y'}{y} &= \ln(\ln x) + x \frac{1}{\ln x} \frac{1}{x} \end{aligned}$$

Multiply both sides by y to get y'

$$\begin{aligned} y' &= y \left[\ln(\ln x) + \frac{1}{\ln x} \right] \\ &= [(\ln x)^x] \left[\ln(\ln x) + \frac{1}{\ln x} \right] \end{aligned}$$

Question 4. [4 Marks] Find the absolute maximum and minimum values of the following function on the given interval.

$$f(x) = x^3 + 3x^2 + 1 \quad -3 \leq x \leq 2$$

Solution: First, $f'(x) = 3x^2 + 6x = 3x(x+2)$ [1 mark]. To determine the critical points, let $f'(x) = 3x(x+2) = 0$, so, the critical points are

$$x = 0 \text{ OR } x = -2 [1 \text{ mark}]$$

The values of f at critical points and end points are:

$$f(0) = 1, \quad f(-2) = 5, \quad f(-3) = 1, \quad f(2) = 21 [1 \text{ mark}]$$

Therefore, the absolute minimum value of f on $-3 \leq x \leq 2$ is $f(0) = f(-3) = 1$ [0.5 mark].

The absolute maximum value of f on $-3 \leq x \leq 2$ is $f(2) = 21$ [0.5 mark].