Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

Calculus III for Engineers
MAT 2322A - Fall 2015
Midterm I
Professor: Victor G. LeBlanc
Time limit: 80 minutes. Closed books.

Name: Solutions ID Number:

Instructions

- This exam has 8 pages and you have 80 minutes to complete it.
- This is a closed book exam. Furthermore, all cell phones, pagers or any other electronic or communication devices are forbidden. **The only calculators which are allowed are Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- Read each question carefully before answering.
- Questions 1 to 3 are multiple choice questions. These questions are worth 2 points each and no partial marks are possible. **Please write your answers in the corresponding boxes in the grid below entitled "Answers to multiple choice Qs".**
- Questions 4 to 6 are long answer questions. Questions 4 and 6 are worth 6 marks each, and question 5 is worth 7 marks, so organize your time accordingly. **A correct answer requires a full, clearly-written and detailed solution.** Answer each question in the space provided, using backs of pages or the extra pages at the end if necessary.
- Do not unstaple the test.
- Good luck!

Answers to multiple choice Qs

1	2	3
C	A	E

Grid below is used for grading
(do not write in this grid)

MCQ	4	5	6	Total
/6	/6	/7	/6	/25

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1. Let f, u and v be differentiable functions such that

$$\begin{array}{ll} u(0,0) = 1 & v(0,0) = 2 \\ u_x(0,0) = -3 & u_y(0,0) = 4 \\ v_x(0,0) = 2 & v_y(0,0) = 3 \\ u_y(1,2) = -2 & v_y(1,2) = -5 \\ \vec{\nabla} f(1,2) = 5\vec{\imath} - 3\vec{\jmath} & \vec{\nabla} f(0,0) = \vec{0} \end{array}$$

If $z(x,y) = f(u(x,y), v(x,y))$, then what is the value of $\dfrac{\partial z}{\partial y}(0,0)$?

A. 0

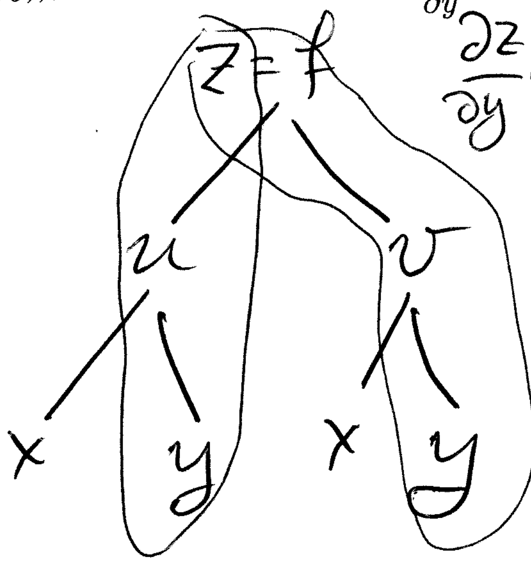
B. -27

C. 11

D. 1

E. 5

F. -21



$$\frac{\partial z}{\partial y}(0,0) = \frac{\partial f}{\partial u}(u(0,0), v(0,0))\frac{\partial u}{\partial y}(0,0) + \frac{\partial f}{\partial v}(u(0,0), v(0,0))\frac{\partial v}{\partial y}(0,0)$$

$$= \frac{\partial f}{\partial u}(1,2)\frac{\partial u}{\partial y}(0,0) + \frac{\partial f}{\partial v}(1,2)\frac{\partial v}{\partial y}(0,0) =$$

$$5 \cdot 4 + (-3) \cdot 3 = 20 - 9 = 11$$

2. If R is the rectangle

$$R = \{ (x,y) \mid 1 \le x \le 3, 0 \le y \le 1 \},$$

what is the value of the double integral $\iint_R (x^2y + x + y)\, dA$?

A. $\dfrac{28}{3}$

B. $\dfrac{29}{3}$

C. 10

D. $\dfrac{31}{3}$

E. $\dfrac{32}{3}$

F. 11

$$\int_1^3 \int_0^1 (x^2y + x + y)\, dy\, dx = \int_1^3 \left(\frac{x^2y^2}{2} + xy + \frac{y^2}{2} \Big|_0^1 \right) dx =$$

$$\int_1^3 \left(\frac{x^2}{2} + x + \frac{1}{2} \right) dx = \frac{x^3}{6} + \frac{x^2}{2} + \frac{x}{2} \Big|_1^3 = \left(\frac{3^3}{6} + \frac{3^2}{2} + \frac{3}{2} \right) - \left(\frac{1}{6} + \frac{1}{2} + \frac{1}{2} \right)$$

$$= \frac{28}{3}$$

3. Let f be a differentiable function such that $\vec\nabla f(0,0) = \vec\imath - \vec\jmath$. If $\vec u$ is a unit vector, then what is the largest possible value that the directional derivative $D_{\vec u} f(0,0)$ can achieve?

A. 0

B. 1

C. 2

D. $\frac{1}{2}$

E. $\sqrt{2}$

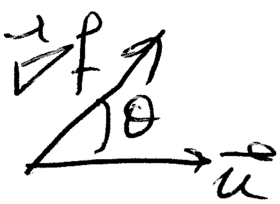
F. -1

As seen in class

$$D_{\vec u} f(0,0) = \vec\nabla f(0,0) \cdot \vec u$$

$$= \|\vec\nabla f(0,0)\| \cdot \underbrace{\|\vec u\|}_{=1} \cdot \cos\theta$$

$$= \|\vec\nabla f(0,0)\| \cos\theta$$



Maximum value is

$\|\vec\nabla f(0,0)\|$ achieved when $\theta = 0$

$$\Longrightarrow \|\vec\nabla f(0,0)\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

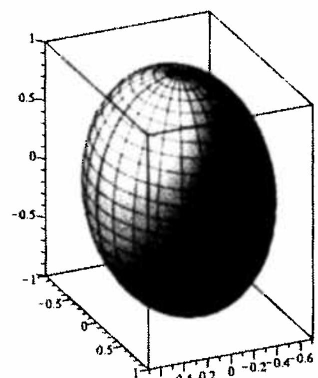
Alternate solution

4. The ellipsoid $2x^2 + y^2 + z^2 = 1$ is illustrated below. Compute the equation of the tangent plane to this surface at the point $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$.

The indicated point is on the graph of the function

$$z = \sqrt{1 - 2x^2 - y^2} = f(x,y)$$

So the tangent plane equation is



$$z = f\left(\tfrac{1}{2}, -\tfrac{1}{2}\right) + f_x\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\left(x - \tfrac{1}{2}\right) + f_y\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\left(y + \tfrac{1}{2}\right)$$

Now $f\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}$, $f_x = \frac{1}{2}(1 - 2x^2 - y^2)^{-\frac{1}{2}} \cdot (-4x)$

$$f_y = \frac{1}{2}(1 - 2x^2 - y^2)^{-\frac{1}{2}} \cdot (-2y)$$

$f_x\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{1}{2}} \cdot (-2) = -2$ $\quad f_y\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{1}{2}}(+1) = +1$

$$\Rightarrow z = \frac{1}{2} + (-2)\left(x - \tfrac{1}{2}\right) + (1)\left(y + \tfrac{1}{2}\right) \Rightarrow$$

$$z = -2x + y + \tfrac{1}{2} + 1 + \tfrac{1}{2} \Rightarrow$$

$$z = -2x + y + 2$$

5. Compute the global extrema of the function

$$f(x,y) = (x-1)^2 + (y+1)^2$$

on the disk

$$S = \{\, (x,y) \,|\, x^2 + y^2 \le 8 \,\}.$$

You must give the (x,y) coordinates AND the value of the function f for each of these global extrema. Also, part of your solution MUST use the **method of Lagrange multipliers**.

Critical points of f: $f_x = 2(x-1)$, $f_y = 2(y+1) \Rightarrow (1,-1)$ is the only critical point of f, and it belongs to S, so we keep it.

Boundary analysis: The boundary is $g(x,y) = x^2+y^2 = 8$. So extreme values of f on the boundary satisfy

$$\vec{\nabla} f = \lambda \vec{\nabla} g, \quad g = 8$$

$$\boxed{\begin{array}{c} 2(x-1) = 2\lambda x \\ 2(y+1) = 2\lambda y \\ x^2+y^2 = 8 \end{array}} \quad \begin{array}{l} \to x(1-\lambda) = 1 \Rightarrow x = \frac{1}{1-\lambda} \\ \to y(1-\lambda) = -1 \Rightarrow y = -\frac{1}{1-\lambda} \end{array}$$

$$\frac{1}{(1-\lambda)^2} + \frac{(-1)^2}{(1-\lambda)^2} = 8 \Rightarrow \frac{2}{(1-\lambda)^2} = 8$$

$$\Rightarrow \frac{1}{4} = (1-\lambda)^2 \Rightarrow 1-\lambda = \pm\frac{1}{2} \Rightarrow \lambda = \frac{3}{2} \text{ or } \lambda = \frac{1}{2}$$

$$\lambda = \frac{3}{2} \Rightarrow x = \frac{1}{1-\frac{3}{2}} = -2, \quad y = \frac{-1}{1-\frac{3}{2}} = 2$$

$$\lambda = \frac{1}{2} \Rightarrow x = \frac{1}{1-\frac{1}{2}} = 2, \quad y = \frac{-1}{1-\frac{1}{2}} = -2$$

	(x,y)	f	
C.P.	$(1,-1)$	0	← global min
	$(2,-2)$	2	
	$(-2,2)$	18	← global max

6. Find and classify all the critical points of the function $f(x,y) = 3x^3 + 3y^2 + 3xy - 1$.

$f_x = 9x^2 + 3y \qquad f_y = 6y + 3x$

$f_y = 0 \Rightarrow x = -2y$

$9\cdot(-2y)^2 + 3y = 0$

$36y^2 + 3y = 0$

$3y(12y+1) = 0 \Rightarrow$

$y = 0$ or $y = -\frac{1}{12}$

$x = 0$ or $x = \frac{1}{6}$

The critical points of f are $(0,0)$ and $(\frac{1}{6}, -\frac{1}{12})$. Now $f_{xx} = 18x$, $f_{xy} = f_{yx} = 3$

$f_{yy} = 6$

$\underline{(0,0)}$: $D = f_{xx} f_{yy} - (f_{xy})^2 = 108x - 9 = 108 \cdot 0 - 9$
$= -9 < 0$

So $(0,0)$ is a saddle

$\underline{(\frac{1}{6}, -\frac{1}{12})}$: $D = 108x - 9 = 108 \cdot \frac{1}{6} - 9 = 18 - 9$
$= 9 > 0$

$\underline{\text{and}}$ $f_{yy} = 6 > 0$ so

$(\frac{1}{6}, -\frac{1}{12})$ is a $\underline{\text{local minimum}}$.

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1. Let f, u and v be differentiable functions such that

$$\begin{array}{ll} u(0,0)=1 & v(0,0)=2 \\ u_x(0,0)=-3 & u_y(0,0)=4 \\ v_x(0,0)=2 & v_y(0,0)=3 \\ u_x(1,2)=-2 & v_x(1,2)=-5 \\ \vec{\nabla} f(1,2)=5\vec{\imath}-3\vec{\jmath} & \vec{\nabla} f(0,0)=\vec{0} \end{array}$$

If $z(x,y)=f(u(x,y),v(x,y))$, then what is the value of $\dfrac{\partial z}{\partial x}(0,0)$?

A. 0

B. -27

C. 11

D. 1

E. 5

F. -21

$$\frac{\partial z}{\partial x}(0,0)=\frac{\partial f}{\partial u}(u(0,0),v(0,0))\frac{\partial u}{\partial x}(0,0)+\frac{\partial f}{\partial v}(u(0,0),v(0,0))\frac{\partial v}{\partial x}(0,0)=$$

$$\frac{\partial f}{\partial u}(1,2)\frac{\partial u}{\partial x}(0,0)+\frac{\partial f}{\partial v}(1,2)\frac{\partial v}{\partial x}(0,0)=$$

$$5\cdot(-3)+(-3)\cdot 2=-15-6$$

$$=-21$$

2. If R is the rectangle

$$R=\{(x,y)\,|\,1\le x\le 3, 0\le y\le 1\},$$

what is the value of the double integral $\iint_R (xy^2+x-y)\,dA$?

A. $\frac{10}{3}$

B. $\frac{21}{3}$

C. 4

D. $\frac{13}{3}$

E. $\frac{14}{3}$

F. 5

$$\int_1^3\int_0^1 (xy^2+x-y)\,dy\,dx=\int_1^3\left(\frac{xy^3}{3}+xy-\frac{y^2}{2}\Big|_0^1\right)dx=$$

$$\int_1^3\left(\frac{x}{3}+x-\frac{1}{2}\right)dx=\frac{x^2}{6}+\frac{x^2}{2}-\frac{x}{2}\Big|_1^3=\left(\frac{3^2}{6}+\frac{3^2}{2}-\frac{3}{2}\right)-\left(\frac{1}{6}+\frac{1}{2}-\frac{1}{2}\right)$$

$$=\frac{13}{3}$$

3. Let f be a differentiable function such that $\vec{\nabla} f(0,0) = 2\vec{\imath} + \vec{\jmath}$. If $\vec{u}$ is a unit vector, then what is the largest possible value that the directional derivative $D_{\vec{u}} f(0,0)$ can achieve?

A. $\sqrt{5}$

B. 1

C. 5

D. 2

E. $\sqrt{2}$

F. -1

As seen in class

$$D_{\vec{u}} f(0,0) = \vec{\nabla} f(0,0) \cdot \vec{u} = \|\vec{\nabla} f(0,0)\| \cdot \underbrace{\|\vec{u}\|}_{=1} \cos\theta$$

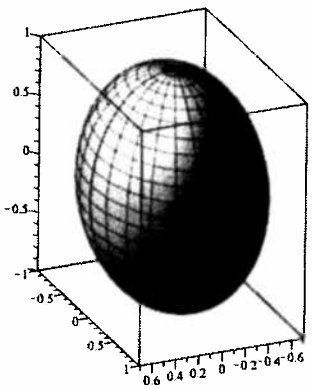
$$= \|\vec{\nabla} f(0,0)\| \cos\theta$$

Maximum value is $\|\vec{\nabla} f(0,0)\|$ achieved when $\theta = 0$

$$\Rightarrow \|\vec{\nabla} f(0,0)\| = \sqrt{2^2+1^2} = \sqrt{5}$$

4. The ellipsoid $x^2 + 2y^2 + z^2 = 1$ is illustrated below. Compute the equation of the tangent plane to this surface at the point $\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

See Sept. 21 lecture for a similar problem solved in class.



The ellipsoid is the level surface $g=1$, where $g(x,y,z) = x^2+2y^2+z^2$. So $\vec{\nabla} g(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is $\perp$ to the ellipsoid, and hence $\perp$ to the tangent plane, i.e. $\vec{\nabla} g(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \cdot \left((x+\frac{1}{2})\vec{i} + (y-\frac{1}{2})\vec{j} + (z-\frac{1}{2})\vec{k}\right) = 0$ is eqn of the tangent plane. Now $\vec{\nabla} g = 2x\vec{i} + 4y\vec{j} + 2z\vec{k}$, so $\vec{\nabla} g(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = -\vec{i} + 2\vec{j} + \vec{k}$. Therefore, the equation of the tangent plane is

$$(-\vec{i} + 2\vec{j} + \vec{k}) \cdot \left((x+\tfrac{1}{2})\vec{i} + (y-\tfrac{1}{2})\vec{j} + (z-\tfrac{1}{2})\vec{k}\right) = 0 \Rightarrow$$

$$-(x+\tfrac{1}{2}) + 2(y-\tfrac{1}{2}) + (z-\tfrac{1}{2}) = 0 \Rightarrow$$

$$-x + 2y + z - \tfrac{1}{2} - 1 - \tfrac{1}{2} = 0 \Rightarrow$$

$$-x + 2y + z - 2 = 0 \Rightarrow$$

$$z = x - 2y + 2$$

5. Compute the global extrema of the function

$$f(x,y) = (x+2)^2 + (y-2)^2$$

on the disk

$$S = \{\,(x,y)\,|\,x^2+y^2 \le 32\,\}.$$

You must give the (x,y) coordinates AND the value of the function f for each of these global extrema. Also, part of your solution MUST use the **method of Lagrange multipliers**.

Critical points of f: $f_x = 2(x+2)$, $f_y = 2(y-2) \Rightarrow (-2,2)$ is the only critical point of f, and it belongs to S, so we keep it.

Boundary analysis: The boundary is $g(x,y) = x^2+y^2 = 32$. So extreme values of f on the boundary satisfy

$$\vec{\nabla} f = \lambda \vec{\nabla} g, \quad g = 32$$

$$\boxed{\begin{array}{l} 2(x+2) = 2\lambda x \\ 2(y-2) = 2\lambda y \\ x^2+y^2 = 32 \end{array}} \qquad \begin{array}{l} x(1-\lambda) = -2 \Rightarrow x = \frac{-2}{1-\lambda} \\ y(1-\lambda) = 2 \Rightarrow y = \frac{2}{1-\lambda} \end{array}$$

$$\frac{4}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 32 \Rightarrow \frac{8}{(1-\lambda)^2} = 32$$

$$\frac{1}{4} = (1-\lambda)^2 \Rightarrow 1-\lambda = \pm\frac{1}{2} \Rightarrow \lambda = 3/2 \text{ or } \lambda = 1/2$$

$$\lambda = 3/2 \Rightarrow x = \frac{-2}{1-3/2} = 4, \; y = \frac{2}{1-3/2} = -4$$

$$\lambda = \frac{1}{2} \Rightarrow x = \frac{-2}{1-1/2} = -4, \; y = \frac{2}{1-1/2} = 4$$

(x,y)	f	
C.P. $(-2,2)$	0	← global min
$(4,-4)$	72	← global max.
$(-4,4)$	8	

6. Find and classify all the critical points of the function $f(x,y) = 2x^2 + 2y^3 + 2xy + 1$.

$f_x = 4x + 2y$ $\quad f_y = 6y^2 + 2x$

$f_x = 0 \Rightarrow y = -2x$ $\quad 6(-2x)^2 + 2x = 0$

$\Rightarrow 24x^2 + 2x = 0$

$2x(12x+1) = 0 \Rightarrow x = -\frac{1}{12}$ or $x=0$

$y = 1/6$ or $y = 0$

The critical points of f are $(0,0)$ and $(-\frac{1}{12}, 1/6)$. Now $f_{xx} = 4$, $f_{xy} = f_{yx} = 2$

$f_{yy} = 12y$

$\underline{(0,0)}$: $D = f_{xx} f_{yy} - (f_{xy})^2 = 48y - (2)^2 = 48 \cdot 0 - 4$

$= -4 < 0$

So $(0,0)$ is a saddle.

$\underline{(-\frac{1}{12}, \frac{1}{6})}$: $D = 48y - 4 = 48 \cdot \frac{1}{6} - 4 = 8 - 4 = 4 > 0$

$\underline{\text{and}}$ $f_{xx} = 4 > 0$ so

$(-\frac{1}{12}, \frac{1}{6})$ is a $\underline{\text{local minimum}}$.