Statistical Science 1024a(W2014)

Ch.3 The Normal Distributions

Note: TABLE A (Standard Normal proportions) is posted in the Resources folder in "Course formula and Tables" posting

PRINTcopy of TABLE A for Ch3 lectures

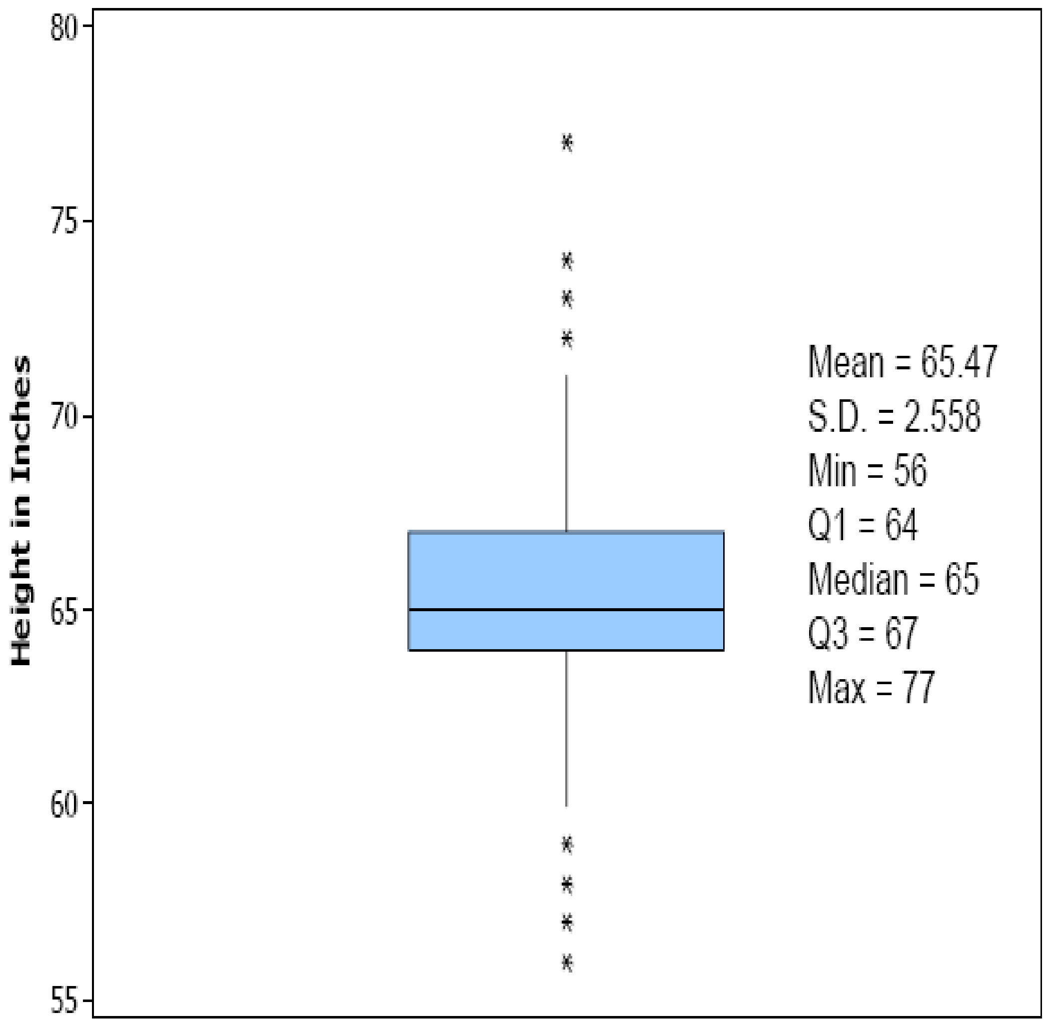
EXPLORING A DISTRIBUTION

1. Always plot your data: make a graph, usually a histogram or a stemplot.
2. Look for the overall pattern (shape, center, spread) and for striking deviations such as outliers.
3. Calculate a numerical summary to briefly describe center and spread.

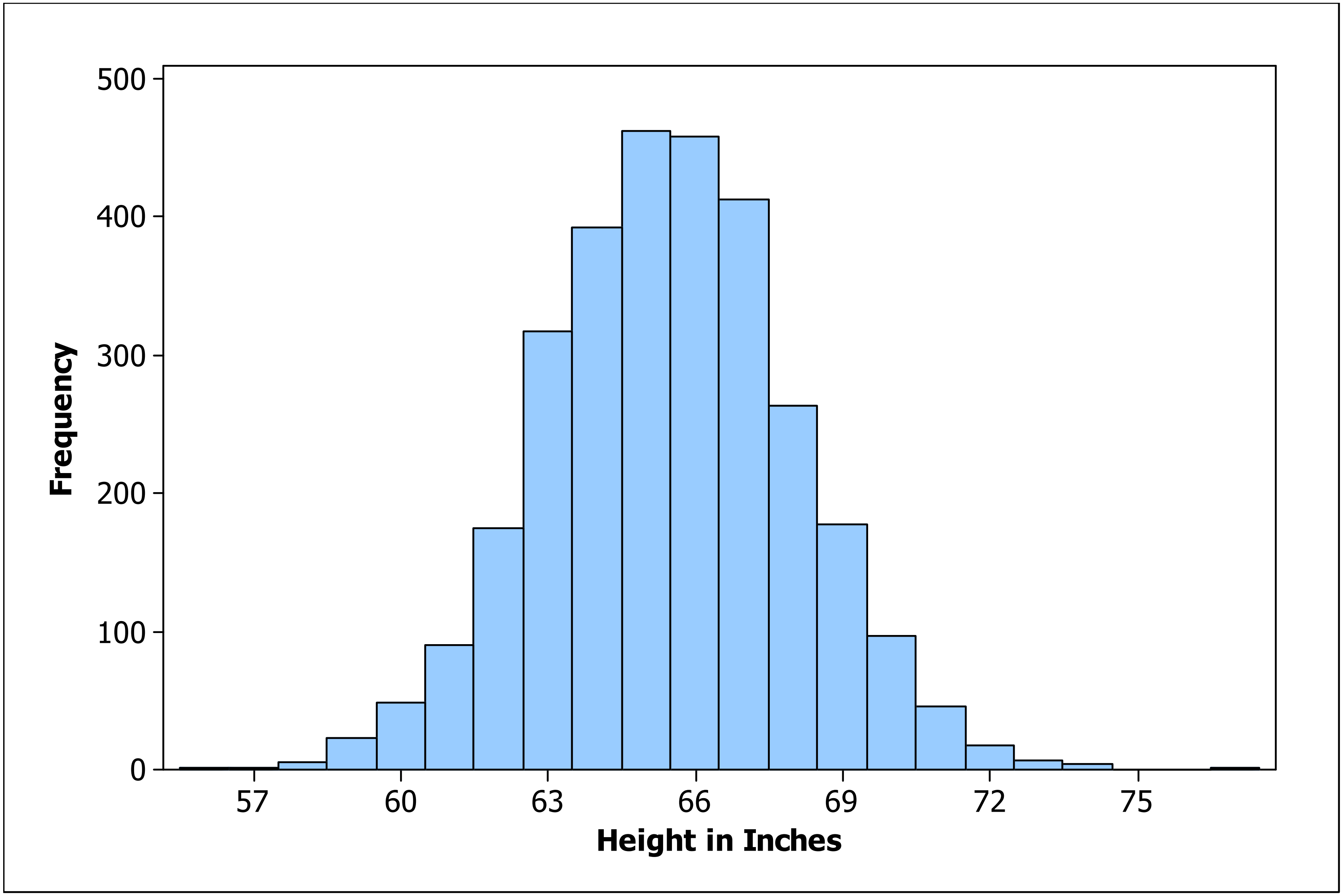
- Sometimes the overall pattern of a large number of observations is so regular that we can describe it by a smooth curve

Example Data: Heights of 3000 Criminals

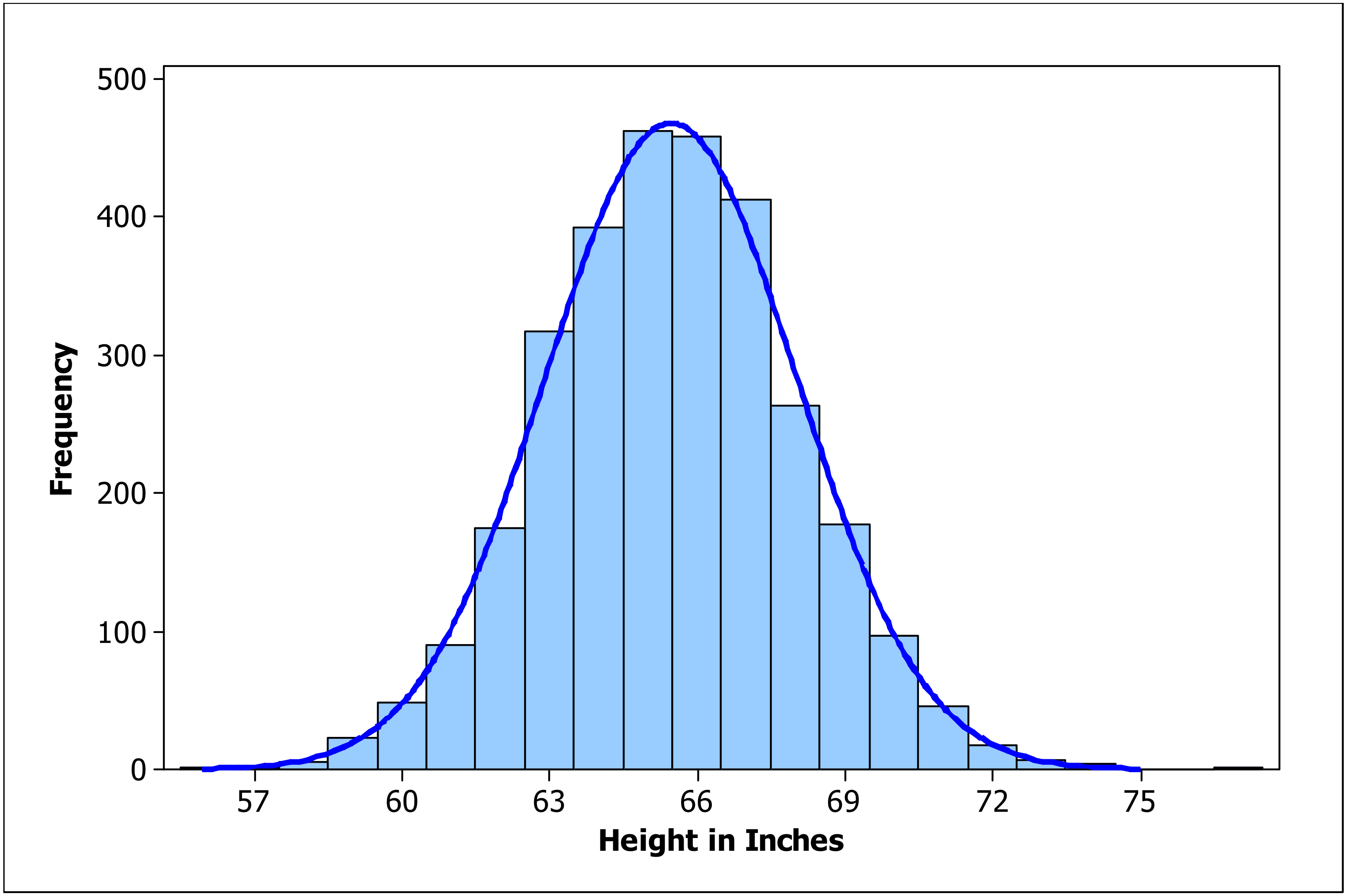
- distribution of heights of 3000 criminals in 1902
- distribution is nearly symmetric
- **(box plot shows outliers)**



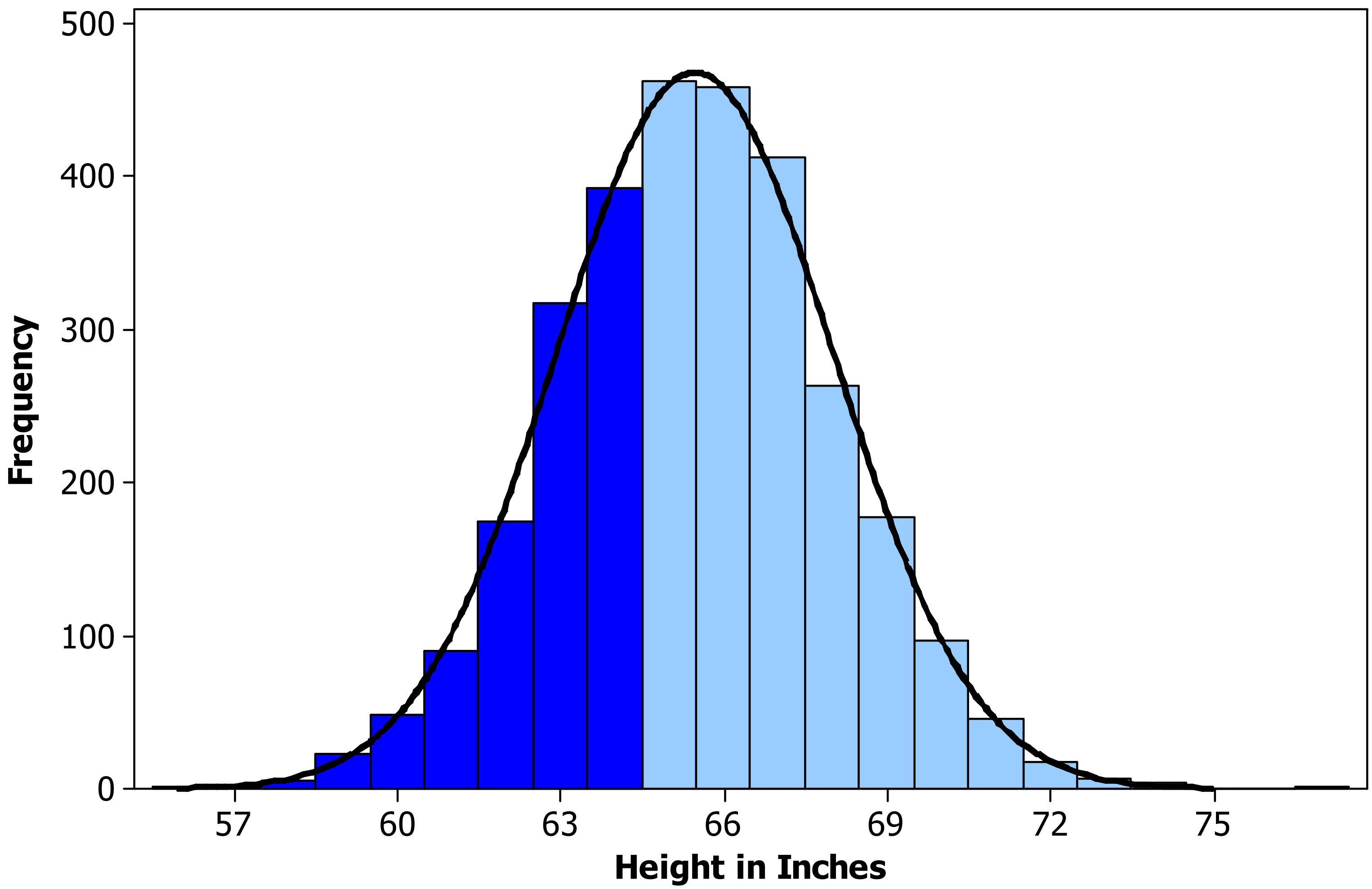
Histogram of Heights



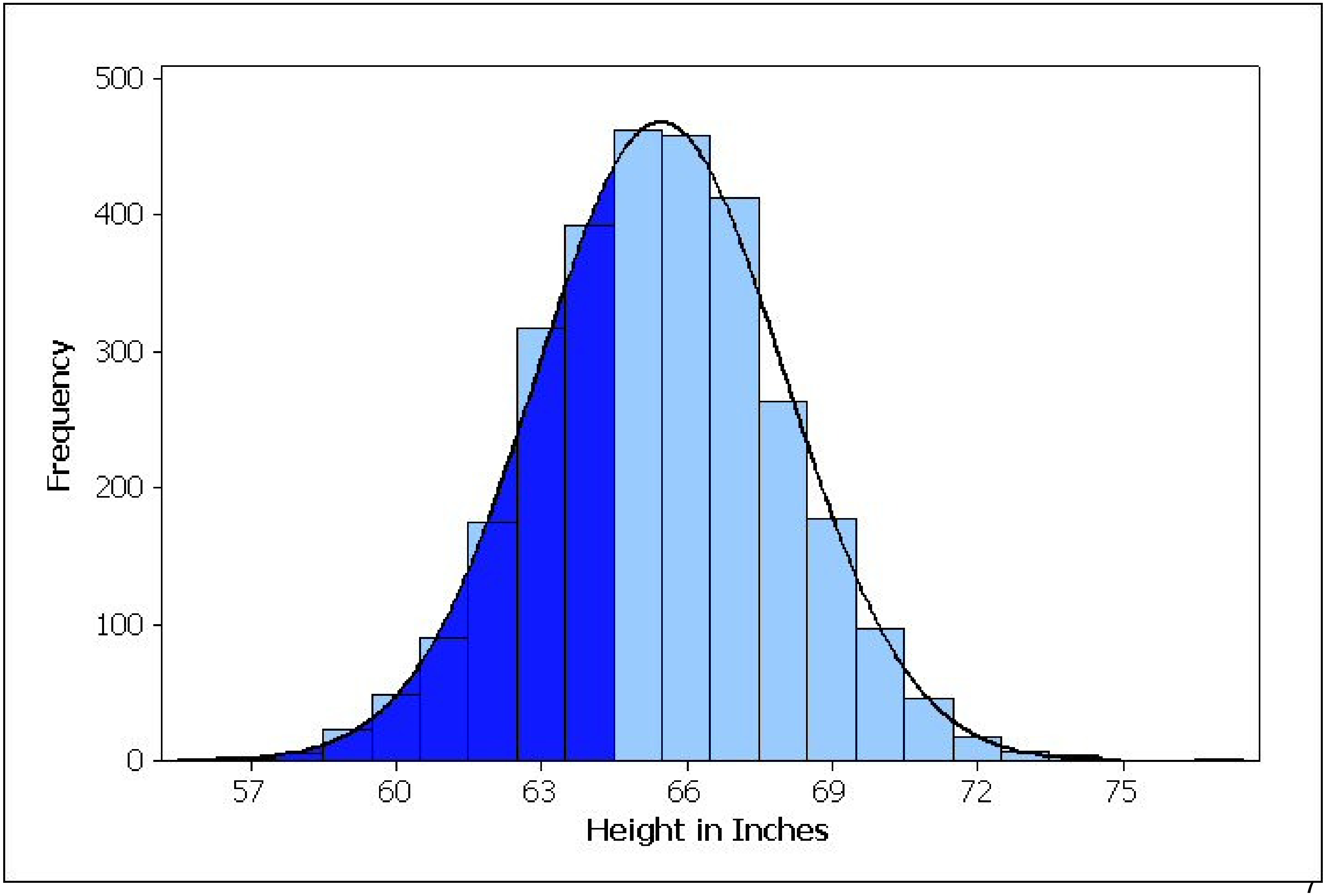
Histogram to Density Curve



Histogram Areas



Density Curve Area



DENSITY CURVE

A **density curve** is a curve that

- is always on or above the horizontal axis, and
- has area exactly 1 underneath it.

A density curve describes the overall pattern of a distribution. The area under the curve and above any range of values is the proportion of all observations that fall in that range.

More on Density Curves

- a density curve describes the theoretical pattern or distribution of the data that can be obtained. The description is in terms of a mathematical function
- for the theory to coincide with the observations, the histogram and the density curve should be similar

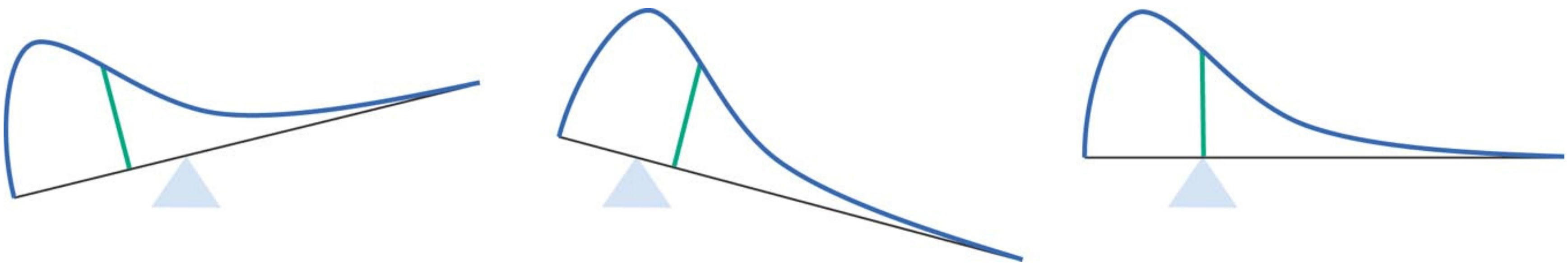
MEDIAN AND MEAN OF A DENSITY CURVE

The **median** of a density curve is the equal-areas point, the point that divides the area under the curve in half.

The **mean** of a density curve is the balance point, at which the curve would balance if made of solid material.

The median and mean are the same for a symmetric density curve. They both lie at the center of the curve. The mean of a skewed curve is pulled away from the median in the direction of the long tail.

The Mean as the Centre of Gravity of the Density Curve



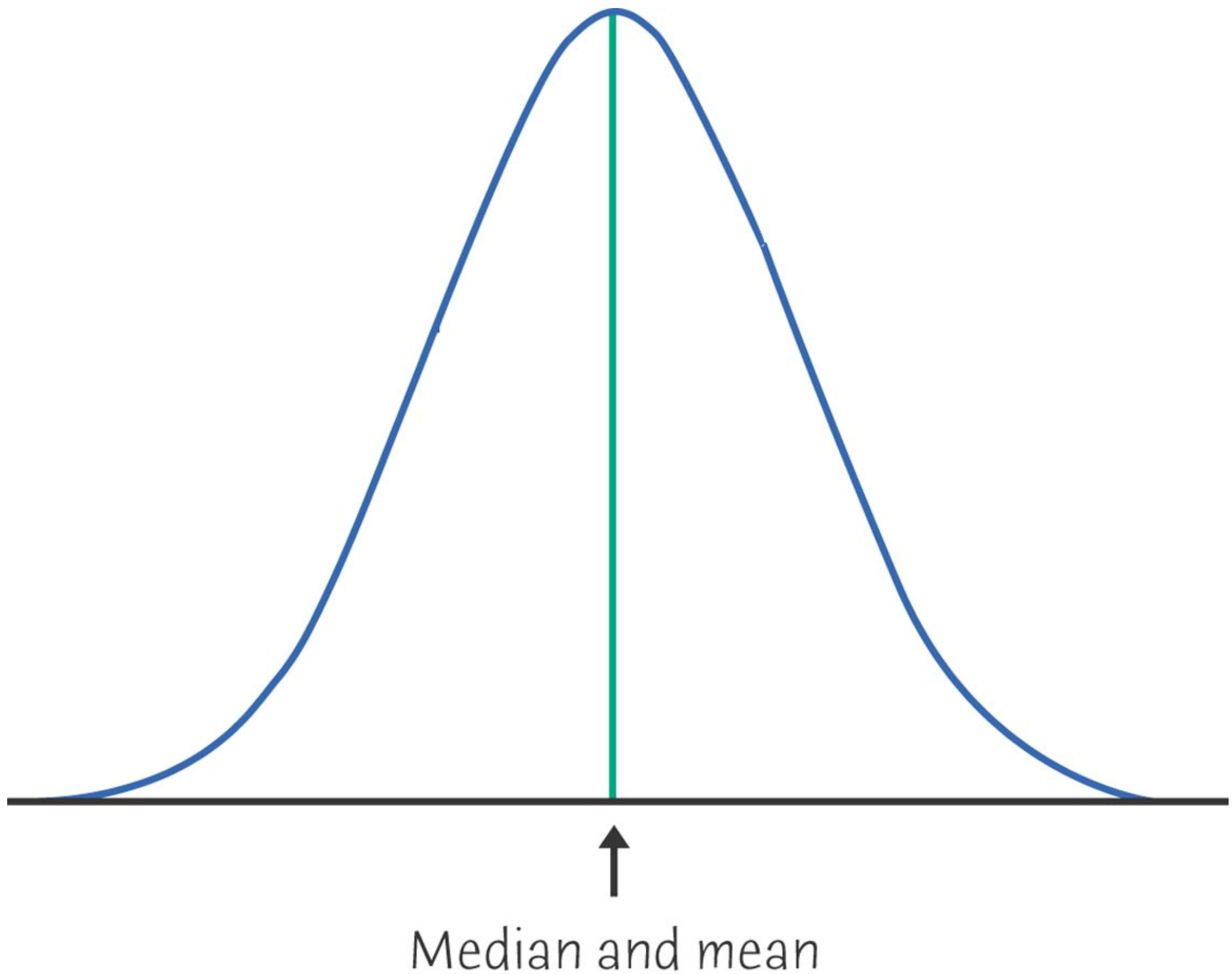
Properties of Means and Medians of Density Curves

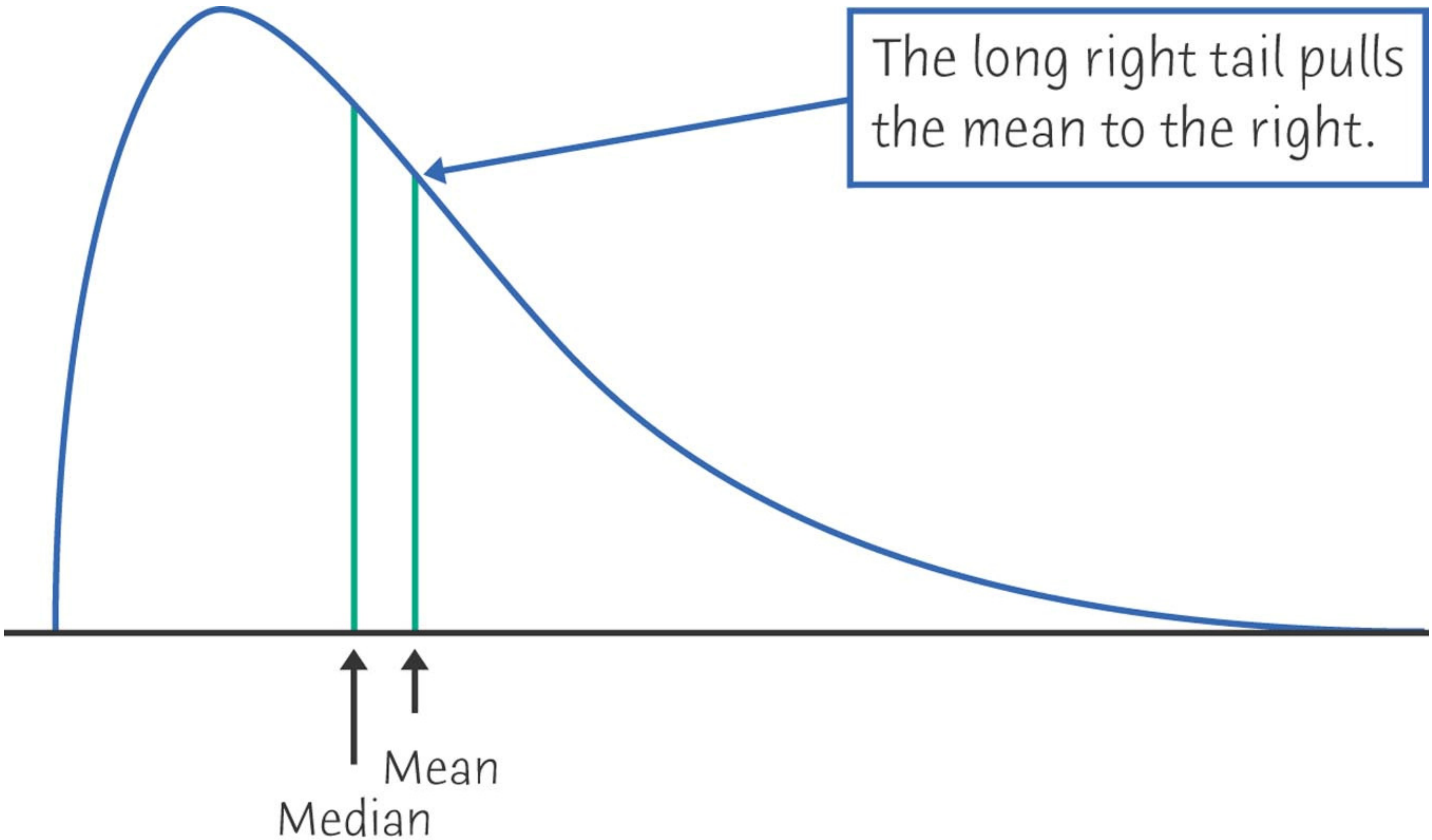
For a symmetric density curve

- mean = median

For an asymmetric or skewed density curve

- mean > median if the long tail of the curve is in the right of the distribution (skewed to the right)
- mean < median if the long tail of the curve is in the left of the distribution (skewed to the left)


Median and mean


The long right tail pulls the mean to the right.
Mean
Median

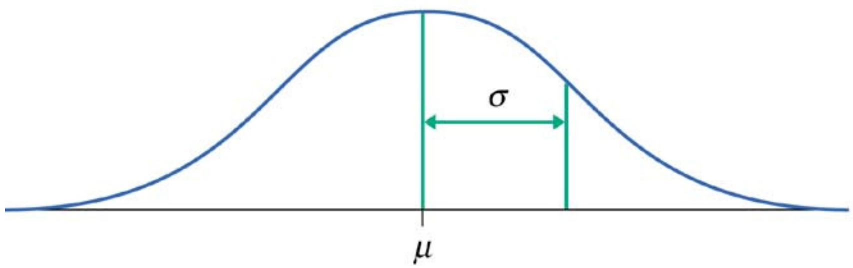
Normal Density Curves

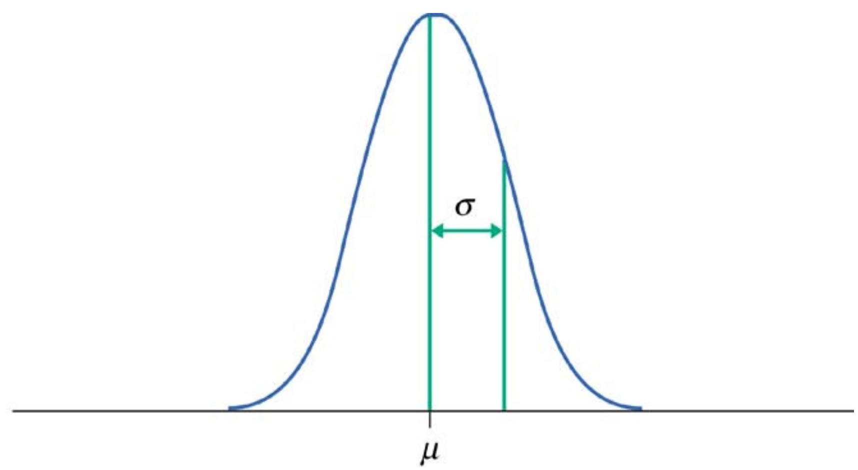
- **Normal Density curves are an important/special class of density curves** for several reasons;
  - They are a good description for some distribtuions of real data
  - They can be used to get a good approximation to results of many kinds of chance outcomes (e.g. repeated coin tosses and outcome probabilities)
  - **Many statistical inference procedures based on Normal distributions work well for roughly symmetric distributions**
    - We will use Normal density curve frequently, especially the **standard normal density curve**

NORMAL DISTRIBUTIONS

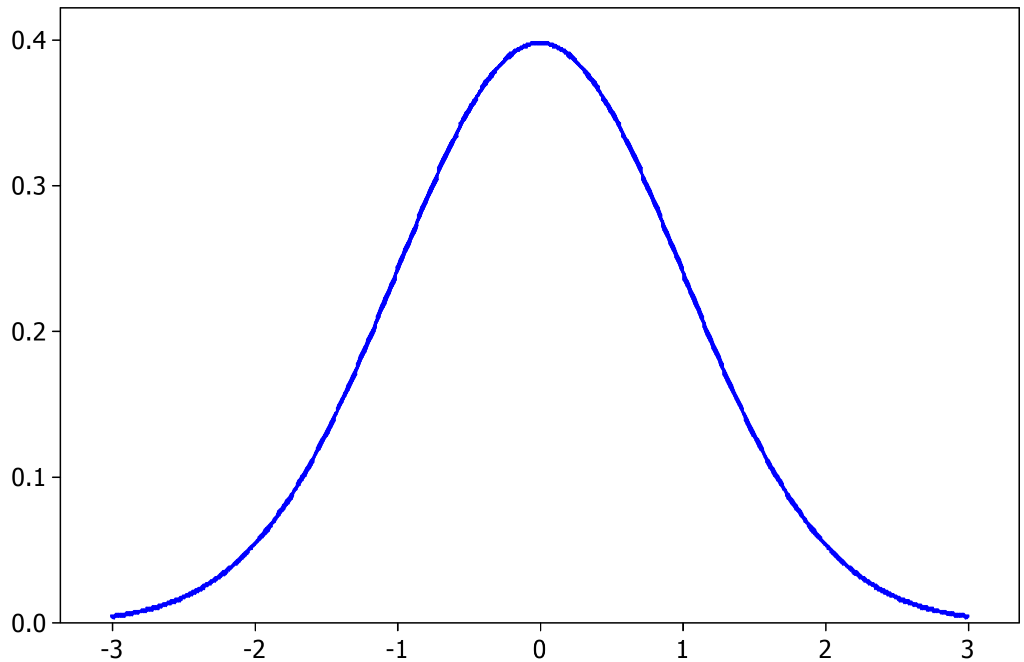
A **Normal distribution** is described by a Normal density curve. Any particular Normal distribution is completely specified by two numbers, its mean and standard deviation.

The mean of a Normal distribution is at the center of the symmetric Normal curve. The standard deviation is the distance from the center to the change-of-curvature points on either side.


σ
μ


σ
μ

Standard Normal Distribution

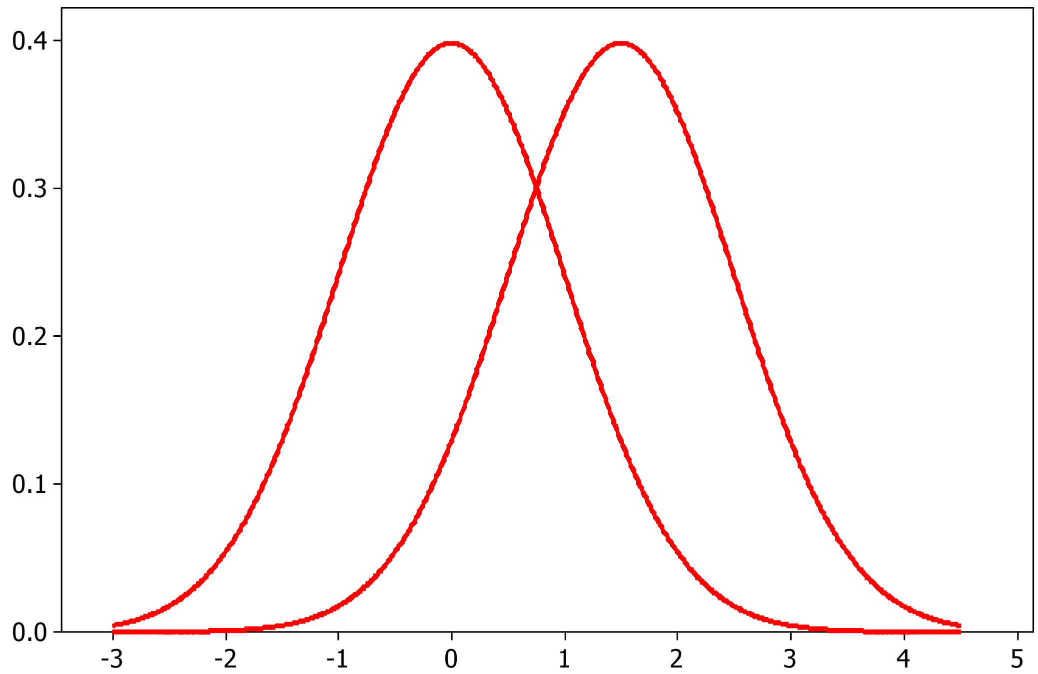


- bell-shaped curve centered around 0 with most of the curve between -3 and +3
- symmetric around 0 so that the **mean and median are both 0**
- **standard deviation equals 1**

(General) Normal Distribution

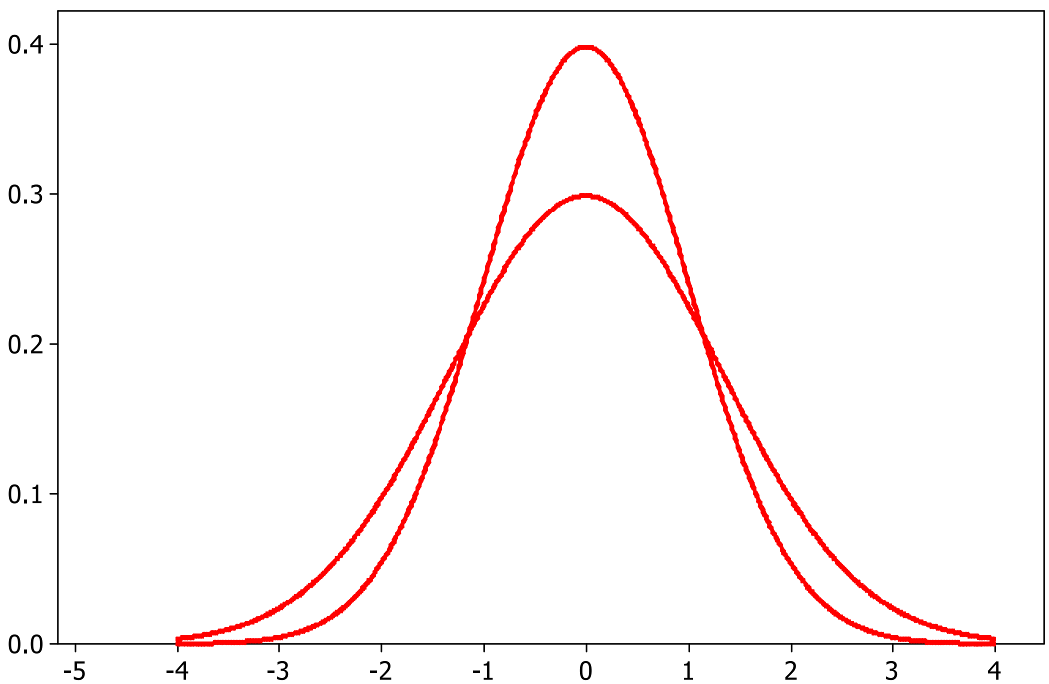
- the **mean** μ of the distribution determines the location of the distribution or the centre of the bell-shaped curve
- the **standard deviation** σ determines the shape or how spread out the bell shape is
- **Notation used: N** (μ, σ) ,Standard Normal Distribution is N(0,1)

Same Standard Deviations, Different Means



- the curve on the right has a larger mean than the curve on the left
- the amount of the shift is equal to the difference in the means

Same Means, Different Standard Deviations



- the lower curve has a larger standard deviation
- the spread of the curve increases with the standard deviation

THE 68–95–99.7 RULE

In the Normal distribution with mean μ and standard deviation σ:

- Approximately **68%** of the observations fall within σ of the mean μ.
- Approximately **95%** of the observations fall within 2σ of μ.
- Approximately **99.7%** of the observations fall within 3σ of μ.

Same Rule, Different Expression

- 68% of the area under the curve is between $\mu - \sigma$ and $\mu + \sigma$
- 95% of the area under the curve is between $\mu - 2\sigma$ and $\mu + 2\sigma$
- 99.7% of the area under the curve is between $\mu - 3\sigma$ and $\mu + 3\sigma$

Example1(68-95-99.7%) Rule

You are given that set of test scores are approximately normal, with mean(μ)= 65, and standard deviation(σ)=10

(a) **give intervals that contain**

(i) The middle 68% of scores

(ii) The middles 95% of scores

(iii) The middle 99.7% of scores

(b) **What percent of people have scores above 65?**

(c) **What percent of people have scores above 85?**

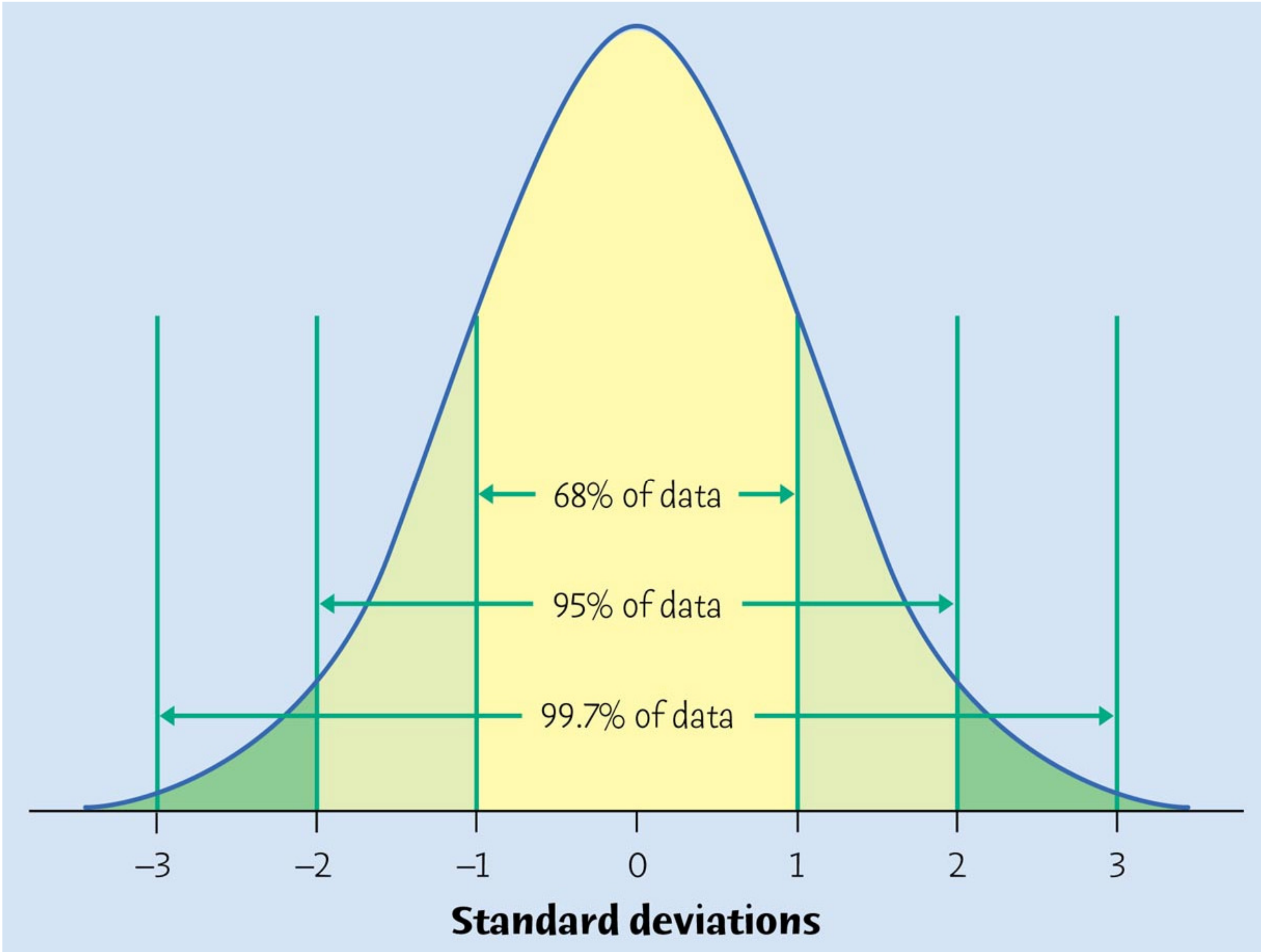
Example2(68-95-99.7 Rule)

If the distribution of ages in this class is normal and an interval containing 68% of the ages is (20,22)

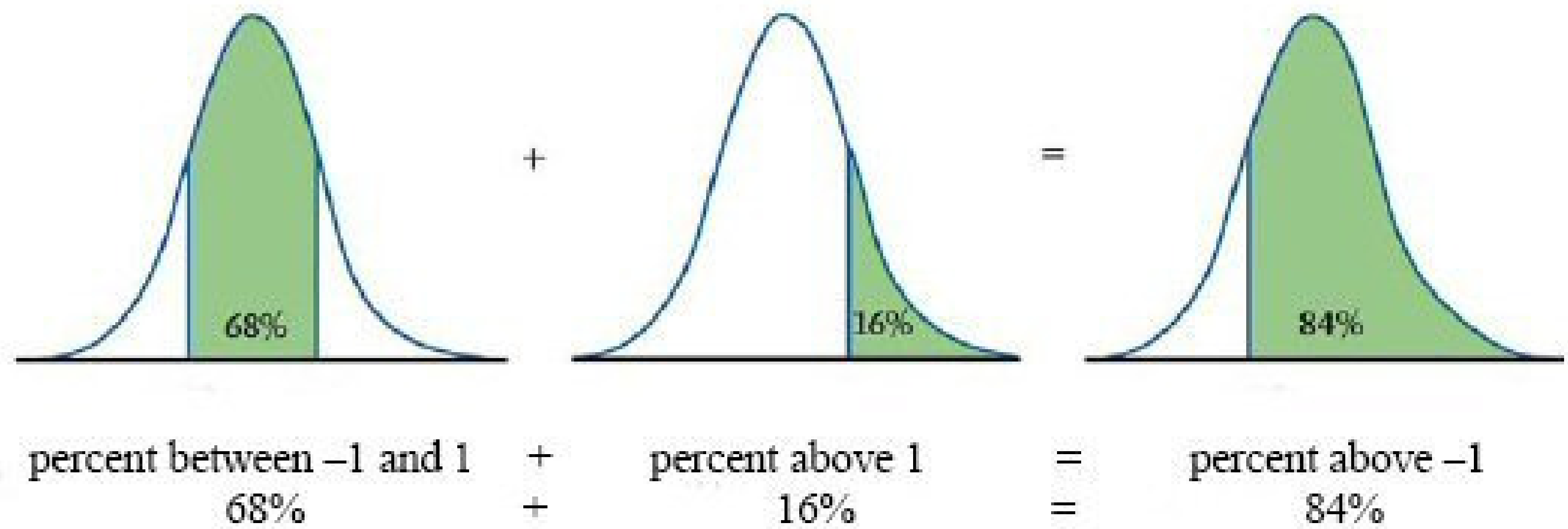
a) What is the mean age?

b) What is the standard deviation of ages?

Standard Normal Distribution



Example Calculation Using the 68-95-99.7 Rule for N(0,1)



STANDARDIZING AND z-SCORES

If x is an observation from a distribution that has mean μ and standard deviation σ, the **standardized value** of x is

$$z = \frac{x - \mu}{\sigma}$$

A standardized value is often called a **z-score.**

Standardizing

$$z = \frac{(x - \mu)}{\sigma},$$

- If $z \geq 0$, x is above the mean, if $z<0$ **,** x is below the mean, if $z=0$, x equals the mean
- z has no units-distance is measured from the mean or in number of standard deviations from the mean

- **Standardizing is useful for**
  - hiding real units such as test scores, salaries, etc
  - Comparison of values from different distributions
  - Determining proportions (percentiles, probabilities)

Examples-Standardizing

- **Ex 1:** Ten years ago, the average price of 3 litre bags of milk was \$3.75 with a standard deviation of 0.5 and the average price of a loaf of bread was \$1.60 with a standard deviation of \$0.40. Suppose in that same year, someone paid \$4.00 for the milk and \$1.75 for the bread. Which was *relatively* more expensive, the bread or the Milk?

- **Ex 2:** Sue is 14 years old and her IQ is 134. The mean IQ score for 14 year olds is 110 and the standard deviation is 16. Sue's brother Mike is 20 years old and his IQ score is 144. The mean IQ score for 20 year olds is 120 and the standard deviation is 20. How would you compare these two IQ scores?

STANDARD NORMAL DISTRIBUTION

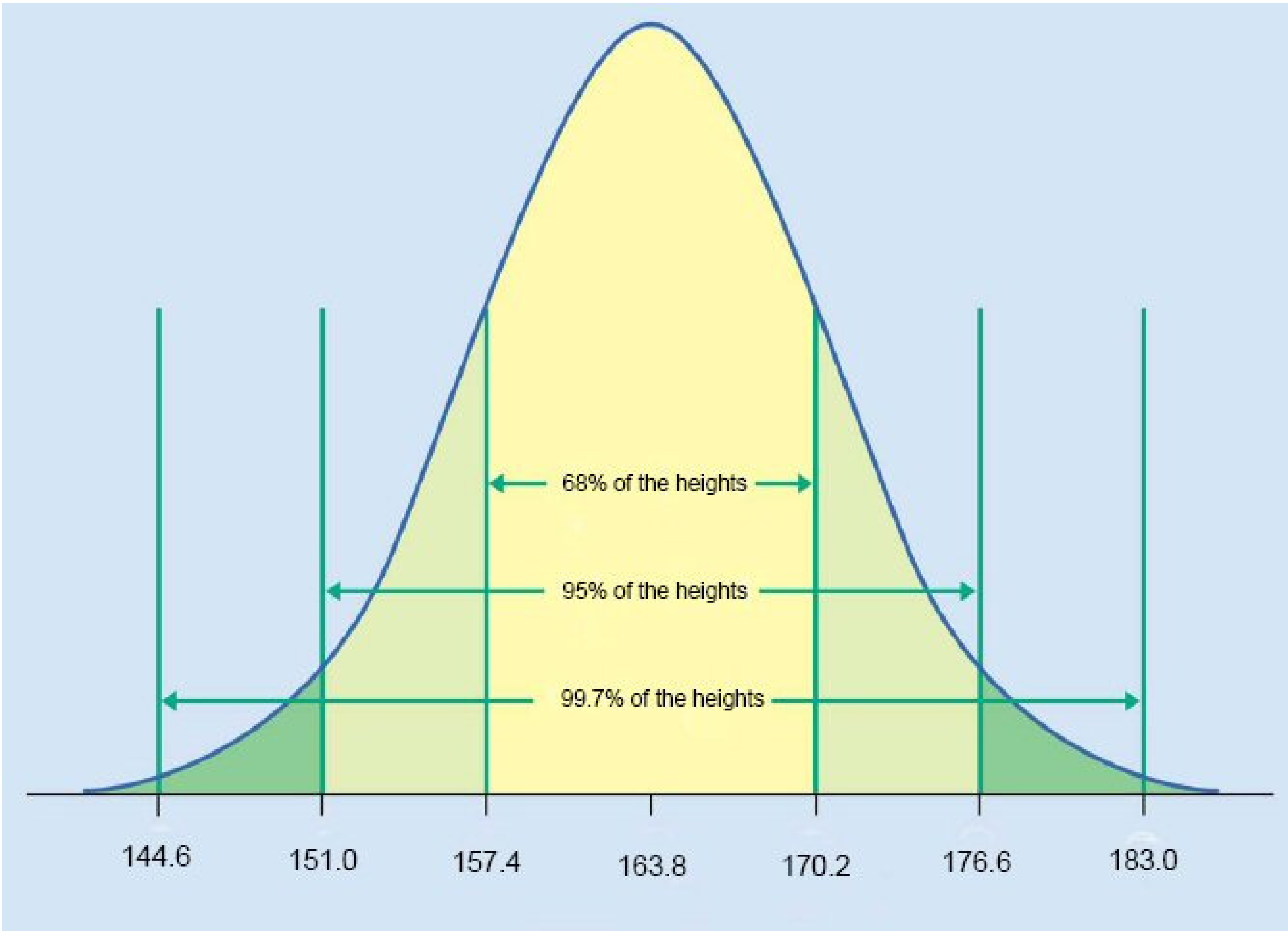
The **standard Normal distribution** is the Normal distribution $N(0, 1)$ with mean 0 and standard deviation 1.

If a variable x has any Normal distribution $N(\mu, \sigma)$ with mean μ and standard deviation σ, then the standardized variable

$$z = \frac{x - \mu}{\sigma}$$

has the standard Normal distribution.

Example: Heights of Women



Assume women's heights have a normal distribution with mean 163.8 cm and a standard deviation of 6.4 cm

Then

$$\frac{\text{height} - 163.8}{6.4}$$

has a standard normal distribution

USING TABLE A TO FIND NORMAL PROPORTIONS

1. State the problem in terms of the observed variable x. **Draw a picture** that shows the proportion you want in terms of cumulative proportions.
2. **Standardize** x to restate the problem in terms of a standard Normal variable z.
3. **Use Table A** and the fact that the total area under the curve is 1 to find the required area under the standard Normal curve.

Table A (copy posted on OWL (see page 1note)

Table A gives the area under the standard Normal curve to the left of any *z*-value.

TABLE A Standard normal probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681

Table for
z-values < 0

Table entry for z is the area under the standard Normal curve to the left of z.

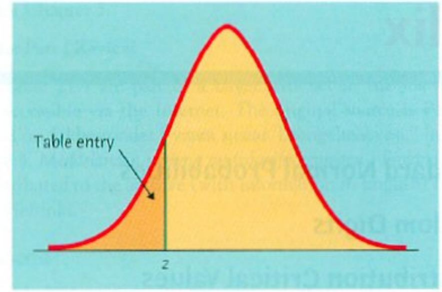


TABLE A Standard Normal probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Table for
z-values > 0

Table entry for z is the area under the standard Normal curve to the left of z.

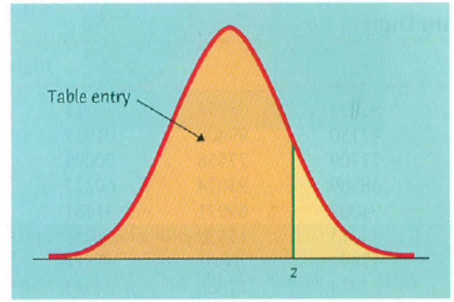
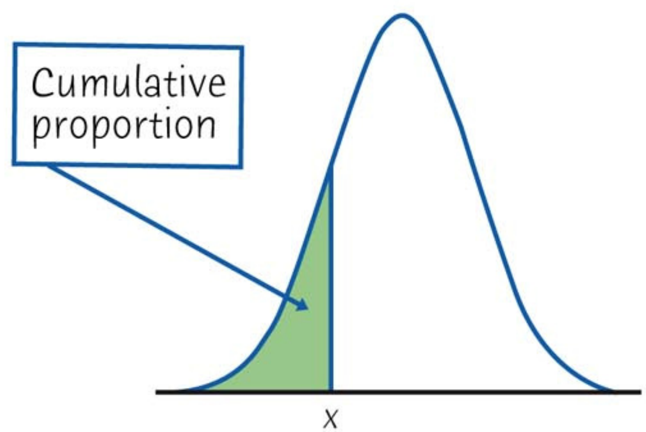


TABLE A Standard Normal probabilities (*continued*)

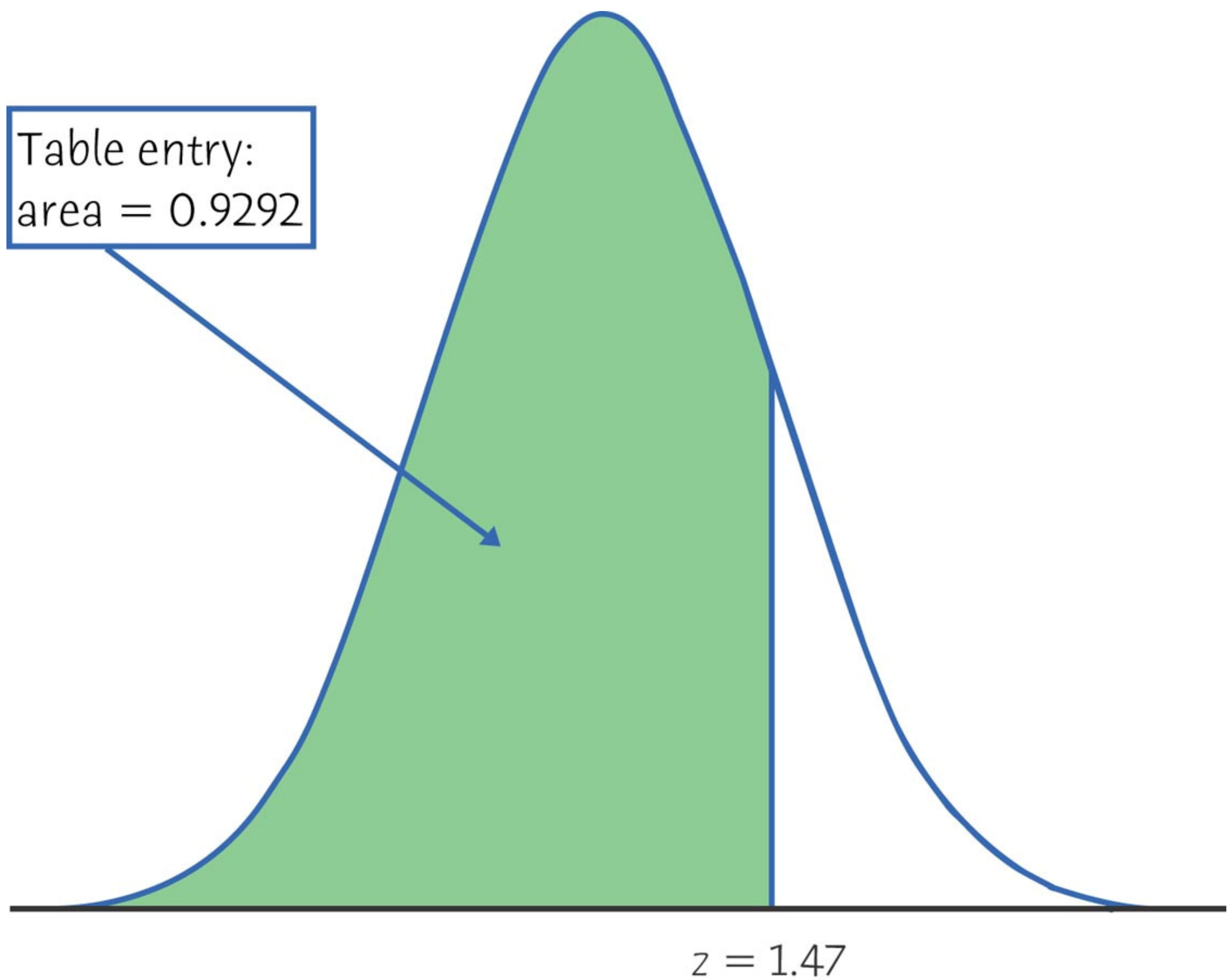
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

CUMULATIVE PROPORTIONS

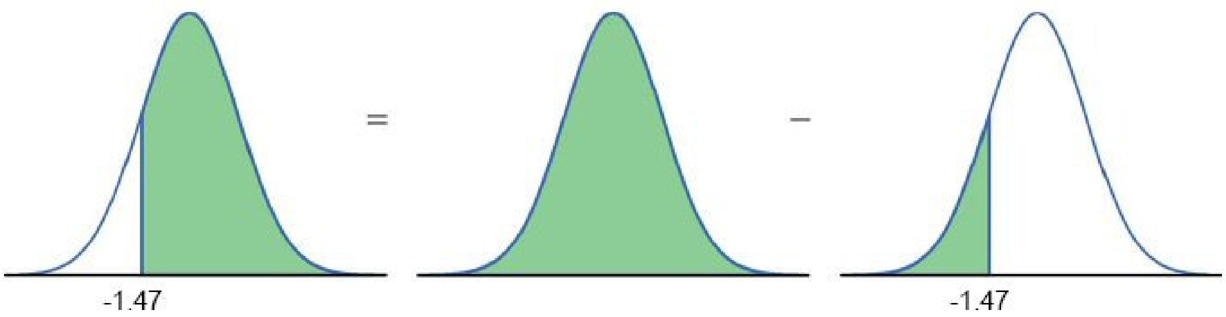
The **cumulative proportion** for a value x in a distribution is the proportion of observations in the distribution that lie at or below x.



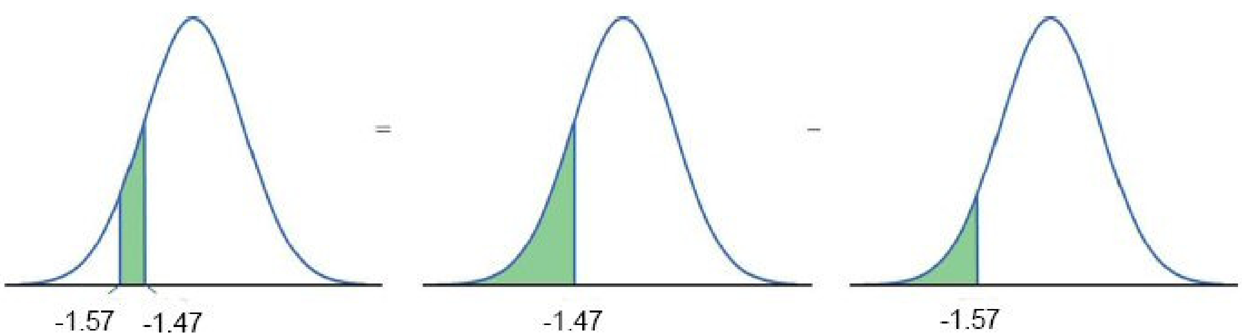
Tables for the Normal Distribution are Based on N(0,1)



Any area can be obtained from the basic table calculation



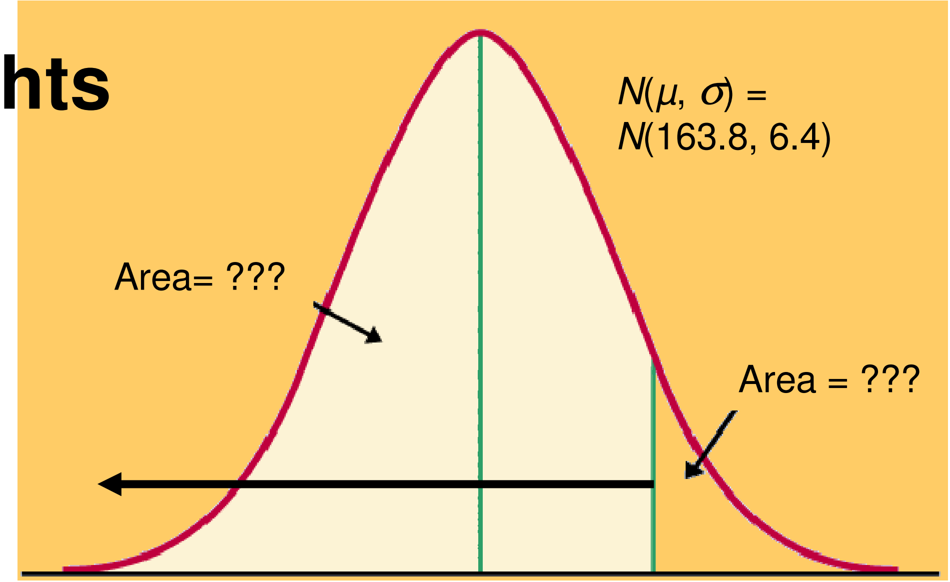
1 – 0.0708
= 0.9292



0.0708 – 0.0582
= 0.0126

Example: Women heights

Women's heights follow the *N*(163.8,6.4) distribution. What percent of women are shorter than 170.2 cm tall?



μ = 163.8 x = 170.2
z = 0 z = 1

mean μ = 163.8 cm
standard deviation σ = 6.4 cm
x (height) = 170.2 cm

We calculate z, the standardized value of x:

$$z = \frac{(x - \mu)}{\sigma}, \quad z = \frac{(170.2 - 163.8)}{6.4} = \frac{6.4}{6.4} = 1 \Rightarrow 1 \text{ stand. dev. from mean}$$

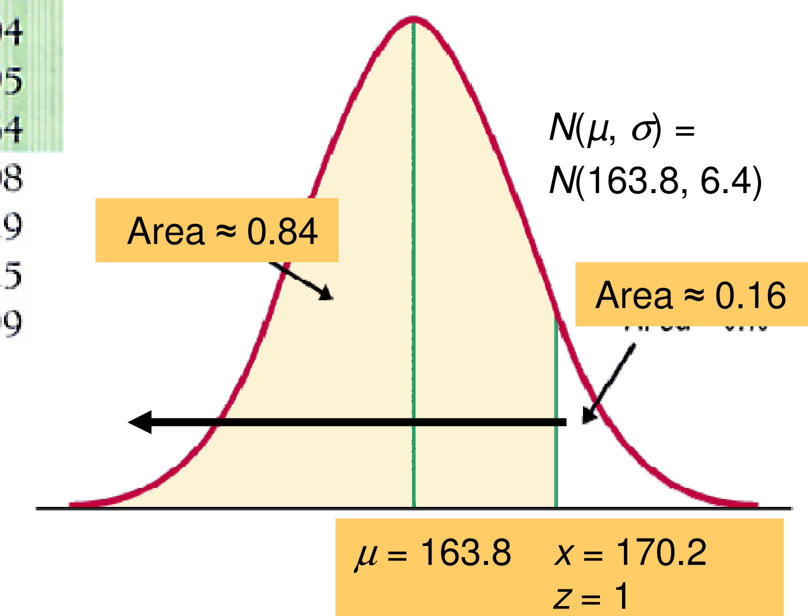
Because of the 68-95-99.7 rule, we can conclude that the percent of women shorter than 170.2 cm should be, approximately, .68 + half of (1 – .68) = .84, or 84%.

Percent of women shorter than 170.2 cm: Exact Calculation

TABLE A Standard normal probabilities (*continued*)

z	.00	.01	.02	.03	.04
0.0	.5000	.5040	.5080	.5120	.5160
0.1	.5398	.5438	.5478	.5517	.5557
0.2	.5793	.5832	.5871	.5910	.5948
0.3	.6179	.6217	.6255	.6293	.6331
0.4	.6554	.6591	.6628	.6664	.6700
0.5	.6915	.6950	.6985	.7019	.7054
0.6	.7257	.7291	.7324	.7357	.7389
0.7	.7580	.7611	.7642	.7673	.7704
0.8	.7881	.7910	.7939	.7967	.7995
0.9	.8159	.8186	.8212	.8238	.8264
1.0	.8413	.8438	.8461	.8485	.8508
1.1	.8643	.8665	.8686	.8708	.8729
1.2	.8849	.8869	.8888	.8907	.8925
1.3	.9032	.9049	.9066	.9082	.9099

For $z = 1.00$, the area under the standard Normal curve to the left of z is 0.8413.



Conclusion:

84.13% of women are <u>shorter</u> than 170.2 cm.

(Also,by subtraction, 1 – 0.8413, or 15.87%, of women are <u>taller</u> than 170.2 cm)

Example-Finding a value

- The finishing times for competitive swimmers in a certain tier on the 100-meter butterfly are normally distributed with a mean of 55 seconds and a standard deviation of 5 seconds.
- What time must a swimmer finish to be in the fastest 3% of finishing times?

Finding a value given a proportion

When you know the proportion, but you don’t know the *x*-value that represents the cut-off, you need to use Table A backward.

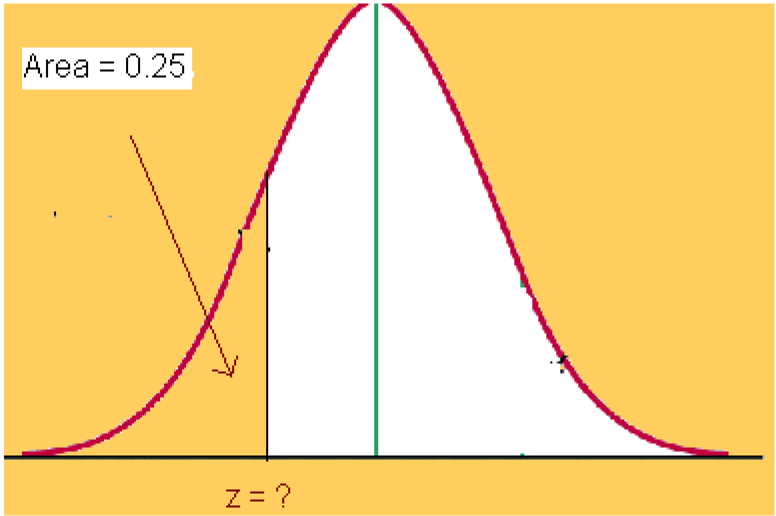
1. State the problem and draw a picture.

2. Use Table A backward, from the inside out to the margins, to find the corresponding *z*.

3. Unstandardize to transform *z* back to the original *x* scale by using the formula:

$$x = \mu + z\sigma$$

Example: Women's heights



Women's heights follow the N(163.8,6.4) distribution. What is the 25th percentile for women's heights?

mean μ = 163.8 cm
standard deviation σ = 6.4 cm
proportion = area under curve = 0.25

We use Table A backward to get the z.

On the left half of Table A (with proportions $\leq$ 0.5), we find that a proportion of 0.25 is between z = -0.67 and –0.68. We'll use z = –0.67.

Now convert back to x:

$$x = \mu + z\sigma = 163.8 + (-0.67)(6.4) = 159.512 \text{ cm}$$

The 25th percentile for women's heights is about 159.5 cm

Other Densities: Example
Uniform Distribution

Suppose that the time it takes a student to eat their lunch is uniformly distributed between 5 and 20 minutes. What percentage of the time does it take a student more than 10 but less than 18 minutes to eat lunch?