Carleton University

Final
Examination
Fall 2013

<u>DURATION: 3 HOURS</u> No. of students: 163

Department Name & Course Number: **Computer Science COMP 2804A**
Course Instructor: Michiel Smid

Authorized memoranda:
NONE

Students MUST count the number of pages in this examination question paper before beginning to write, and report any discrepancy to the proctor. This question paper has 11 pages (not including the cover page).

This examination question paper MAY be taken from the examination room.

In addition to this question paper, students require:

an examination booklet:	**no**
a Scantron sheet:	**yes**

Answers: 1C, 2C, 3B, 4B, 5A, 6A, 7D, 8C, 9A, 10A, 11D, 12B, 13A, 14C, 15D, 16A, 17B, 18B, 19C, 20D, 21A, 22A, 23B, 24B, 25C, 26A, 27C, 28A, 29A, 30D, 31B, 32B, 33D, 34A

Instructions:

1. This is a closed book exam. No aids, notes, or calculating devices are allowed.
2. All questions must be answered on the scantron sheet.

Marking scheme: Each question is worth 3 marks, except the last one, which is worth 1 mark.

- Newton: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.
- Geometric distribution: Assume an experiment has a success probability of p. We perform the experiment until it is successful for the first time. The expected number of times we perform the experiment is $1/p$.

1. Let A be a set of size 25 and let B be a set of size 17. How many functions $f : A \to B$ are there?

   (a) $17!$
   (b) $\frac{25!}{17!}$
   (c) 17^{25}
   (d) 25^{17}

2. Let A be a set of size 6 and let B be a set of size 25. How many one-to-one functions $f : A \to B$ are there?

   (a) 25^6
   (b) $\frac{25!}{6!}$
   (c) $\frac{25!}{19!}$
   (d) $\frac{25!}{20!}$

3. Let A be a set of size 5 and let B be a set of size 14. How many functions $f : A \to B$ are there that are **not** one-to-one?

   (a) $14^5 - \frac{14!}{8!}$
   (b) $14^5 - \frac{14!}{9!}$
   (c) $\frac{14!}{9!} - 14^5$
   (d) $5^{14} - \frac{14!}{9!}$

4. You are given 6 distinct books and 5 *identical* blocks of wood. How many ways are there to arrange these books and blocks in a straight line?

   (a) $\frac{11!}{4!}$
   (b) $\frac{11!}{5!}$
   (c) $\frac{11!}{6!}$
   (d) $\binom{11}{6}$

5. Let S be a set of size 37, and let x and y be two distinct elements of S. How many subsets of S are there that contain x but do not contain y?

   (a) 2^{35}
   (b) 2^{36}
   (c) $2^{37} - 2^{35}$
   (d) $2^{35} + 2^{36}$

6. How many bitstrings of length 6 start with 10 or end with 01?

   (a) 28
   (b) 30
   (c) 32
   (d) 34

7. The number of different strings that can be made by reordering the 10 letters of the word HELLOHELLO is

   (a) 10!
   (b) 4!2!2!2!
   (c) $\binom{10}{4}\binom{6}{2}\binom{4}{2}\binom{3}{2}$
   (d) none of the above.

8. A jar contains 17 red balls and 22 blue balls. How many ways are there to choose, without replacement, 8 balls from this jar?

   (a) $\binom{17}{8}\binom{22}{8}$
   (b) $\sum_{k=0}^{8}(\binom{17}{k} + \binom{22}{8-k})$
   (c) $\binom{39}{8}$
   (d) $\binom{39}{8}8!$

9. How many bitstrings of length 33 are there that start with 1010, end with 0101, and contain exactly 11 zeros?

   (a) $\binom{25}{7}$
   (b) $\binom{33}{7} - 2^8$
   (c) $\binom{33}{7} - 2 \cdot 2^4$
   (d) none of the above.

10. How many solutions are there to the equation $x_1 + x_2 + x_3 = 16$, where $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ are integers?

    (a) $\binom{18}{2}$
    (b) $\binom{18}{3}$
    (c) $\binom{19}{2}$
    (d) $\binom{19}{3}$

11. What is the coefficient of $x^{111}y^{222}$ in the expansion of $(7x + 14y)^{333}$?

    (a) $\binom{333}{222}(7x)^{222}(14y)^{111}$
    (b) $\binom{333}{222}(7x)^{111}(14y)^{222}$
    (c) $\binom{333}{222}7^{222}14^{111}$
    (d) $\binom{333}{222}7^{111}14^{222}$

12. Consider the following recursive algorithm FIB, which takes as input an integer $n \geq 0$:

    Algorithm FIB(n):
    if $n = 0$ or $n = 1$
    then $f = n$
    else $f =$ FIB($n - 1$) + FIB($n - 2$)
    endif;
    return f

    When running FIB(7), how many calls are there to FIB(3)?

    (a) 4
    (b) 5
    (c) 6
    (d) 7

13. The Fibonacci numbers are defined as follows: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.
    Consider again the recursive algorithm FIB, which takes as input an integer $n \geq 0$:

    Algorithm FIB(n):
    if $n = 0$ or $n = 1$
    then $f = n$
    else $f =$ FIB($n - 1$) + FIB($n - 2$)
    endif;
    return f

    For $n \geq 3$, run algorithm FIB(n) and let a_n be the number of times that FIB(1) is called. Which of the following is true?

    (a) for all $n \geq 3$, $a_n = f_n$
    (b) for all $n \geq 3$, $a_n = f_{n-1}$
    (c) for all $n \geq 3$, $a_n = f_{n+1}$
    (d) for all $n \geq 3$, $a_n = f_n + 1$

14. The Fibonacci numbers are defined as follows: $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_{n-1} + f_n$ for $n \geq 1$. Which of the following is true?

    (a) for all $n \geq 1$: $f_1^2 + f_2^2 + f_3^2 + \ldots + f_n^2 = (f_n)^2$
    (b) for all $n \geq 1$: $f_1^2 + f_2^2 + f_3^2 + \ldots + f_n^2 = f_{n-1} f_n$
    (c) for all $n \geq 1$: $f_1^2 + f_2^2 + f_3^2 + \ldots + f_n^2 = f_n f_{n+1}$
    (d) none of the above

15. Consider the following recursive function:

$$\begin{aligned} f(0) &= 7, \\ f(n+1) &= f(n) + 6n + 1 \text{ for all integers } n \geq 0. \end{aligned}$$

    Which of the following is true?

    (a) for all $n \geq 0$: $f(n) = 2n^2 + 2n + 7$
    (b) for all $n \geq 0$: $f(n) = 3n^2 + 2n + 7$
    (c) for all $n \geq 0$: $f(n) = 2n^2 - 2n + 7$
    (d) for all $n \geq 0$: $f(n) = 3n^2 - 2n + 7$

16. Let $A = \{1, 2, 3, \ldots, 100\}$. We choose a uniform random element x in A and, independently, choose a uniform random element y in A. What is the probability that $x = y$?

    (a) $\frac{1}{100}$
    (b) $\frac{1}{100 \cdot 99}$
    (c) $\frac{1}{100 \cdot 100}$
    (d) $\frac{1}{\binom{100}{2}}$

17. We flip a fair coin (independently) 100 times. What is the probability that the coin comes up heads exactly 37 times?

    (a) $\binom{100}{37}(1/2)^{37}$
    (b) $\binom{100}{37}(1/2)^{100}$
    (c) $\frac{1}{\binom{100}{37}}$
    (d) $\binom{100}{37}$

18. Assume you answer the first question in this exam by choosing one of the four answers uniformly at random. You answer the second question by choosing, again uniformly at random, one of the three answers you did not choose in the first question. What is the probability that you answer the second question correctly?

    (a) $\frac{1}{3}$
    (b) $\frac{1}{4}$
    (c) $\frac{1}{4} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{4}$
    (d) none of the above

19. A jar contains 7 red balls and 9 blue balls. We choose, uniformly at random and without replacement, 5 balls. Let A be the event

$$A = \text{“exactly 2 of the balls are red or exactly 3 of the balls are blue”}.$$

    What is $\Pr(A)$?

    (a) $\frac{\binom{7}{2}}{\binom{16}{5}}$
    (b) $\frac{2 \cdot \binom{7}{2}\binom{9}{3}}{\binom{16}{5}}$
    (c) $\frac{\binom{7}{2}\binom{9}{3}}{\binom{16}{5}}$
    (d) $\frac{\binom{7}{2}+\binom{9}{3}}{\binom{16}{5}}$

20. A jar contains 7 red balls and 9 blue balls. We choose, uniformly at random and without replacement, 3 balls. Define the following two events:

$$A = \text{“exactly 2 of the balls are red”}$$

$$B = \text{“the number of red balls is even”}$$

    What is the conditional probability $\Pr(A \mid B)$? (Recall that 0 is even.)

    (a) $\frac{\binom{9}{3}+9\cdot\binom{7}{2}}{9\cdot\binom{7}{2}}$
    (b) $\frac{9\cdot\binom{7}{2}}{\binom{16}{3}}$
    (c) $\frac{\binom{9}{3}+9\cdot\binom{7}{2}}{\binom{16}{3}}$
    (d) $\frac{9\cdot\binom{7}{2}}{\binom{9}{3}+9\cdot\binom{7}{2}}$

21. Assume that a newborn baby is a girl with probability p and a boy with probability $1 - p$. Also assume that the genders of different newborns are independent of each other. Consider a person who has two children. Define the following two events:

$$A = \text{"both children are girls"}$$

$$B = \text{"at least one of the children is a girl"}$$

What is the conditional probability $\Pr(A \mid B)$?

(a) $\frac{p}{2-p}$

(b) $\frac{2-p}{p}$

(c) p

(d) $\frac{1}{p}$

22. We flip a fair coin (independently) three times. Define the following two events:

$$A = \text{"the number of tails is odd"}$$

$$B = \text{"the first coin comes up heads"}$$

True or false: The events A and B are independent.

(a) True

(b) False

23. We flip a fair coin (independently) three times. Define the following two events:

$$A = \text{"the number of tails is odd"}$$

$$B = \text{"the number of heads is even"}$$

True or false: The events A and B are independent. (Recall that 0 is even.)

(a) True

(b) False

24. We flip a fair coin (independently) two times. Define the following three events:

$$A = \text{“the number of heads is odd”}$$

$$B = \text{“the first coin comes up heads”}$$

$$C = \text{“the second coin comes up tails”}$$

True or false: The events A, B, and C are independent.

(a) True
(b) False

25. Let X be a random variable with probability distribution $\Pr(X = 0) = 1/2$, $\Pr(X = 1) = 1/12$, $\Pr(X = 2) = 1/4$, $\Pr(X = 3) = 1/6$. What is the expected value $E(X)$ of X?

(a) 3/2
(b) 12/13
(c) 13/12
(d) 3

26. True or false: If X and Y are random variables, then $E(X + Y) = E(X) + E(Y)$, even if X and Y are not independent.

(a) True
(b) False

27. We flip a fair coin (independently) three times. Let X be the random variable whose value is the number of times the coin comes up heads. What is the expected value $E(X)$ of X?

(a) 1
(b) 5/4
(c) 3/2
(d) 2

28. Consider a coin that comes up heads with probability p and tails with probability $1 - p$. We flip this coin (independently) and stop as soon as it comes up heads for the fifth time. Let X be the random variable whose value is the total number of times we flip the coin. What is the expected value $E(X)$ of X?

    (a) $5/p$
    (b) $5p$
    (c) $1/p^5$
    (d) p^5

29. A jar contains 17 red balls and 5 blue balls. Repeat the following 12 times: Choose one ball uniformly at random (and leave it in the jar). Let X be the random variable whose value is the number of blue balls that we choose. What is the expected value $E(X)$ of X?

    (a) $\frac{30}{11}$
    (b) 6
    (c) $12 \cdot \frac{5}{17}$
    (d) $12 \cdot \frac{17}{5}$

30. A group of $n \geq 3$ people is sitting at a round table, so that each person has two neighbors, one clockwise neighbor and one counter clockwise neighbor. Each person flips a fair and independent coin. A person starts singing if and only if (i) his coin comes up heads, (ii) the coin of his clockwise neighbor comes up tails, and (iii) the coin of his counter clockwise neighbor comes up tails. Let X be the random variable whose value is the number of people that are singing. What is the expected value $E(X)$ of X?

    (a) $n/2$
    (b) $n/3$
    (c) $n/4$
    (d) $n/8$

31. True or false: If X and Y are random variables, then

$$E(\max(X, Y)) = \max(E(X), E(Y)).$$

    (a) True
    (b) False

32. Let K_n be the complete graph on n vertices, in which each pair of vertices is connected by an edge. For each each edge e of K_n, we flip a fair and independent coin; if the coin comes up heads, we color e red, if it comes up tails, we color e blue.
Assume that the vertex set of K_n is $\{1, 2, 3, \ldots, n\}$. Let K' be the subgraph of K_n induced by the vertices $1, 2, \ldots, k$ (i.e., an edge (i, j) of K_n is in K' if and only if both i and j are at most k). Let A be the event

$$A = \text{ “all edges of } K' \text{ have the same color”}$$

What is $\Pr(A)$?

(a) $1/2^{\binom{k}{2}}$
(b) $2/2^{\binom{k}{2}}$
(c) $2^{\binom{k}{2}}/2^{\binom{n}{2}}$
(d) $2 \cdot 2^{\binom{k}{2}}/2^{\binom{n}{2}}$

33. Consider a multiple choice exam with 100 questions, in which for each question, four options are given to choose from. You answer each question by choosing an answer uniformly at random, and independently of the other answers. What is the expected number of correct answers?

(a) $\sum_{k=0}^{100} (1/4)^k (3/4)^{100-k}$
(b) $\sum_{k=0}^{100} k(1/4)^k (3/4)^{100-k}$
(c) $\sum_{k=0}^{100} \binom{100}{k} (1/4)^k (3/4)^{100-k}$
(d) $\sum_{k=0}^{100} k\binom{100}{k} (1/4)^k (3/4)^{100-k}$

34. How do you feel?

(a) I need a beer.