Midterm COMP 2804

October 23, 2013

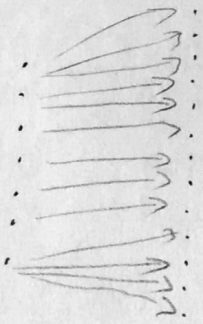
- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.

Marking scheme: Each of the 17 questions is worth 1 mark.

- Newton: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

Solutions:
1C, 2B, 3B, 4A, 5C, 6B, 7D, 8B, 9A, 10D, 11D, 12C, 13B, 14D, 15B, 16A, 17B

1. Let A be a set of size 7 and let B be a set of size 13. How many one-to-one functions $f : A \to B$ are there?



$\frac{n!}{n-m!} = \frac{13!}{(13-7)!} = \frac{13!}{6!}$

(a) $\frac{6!}{13!}$
(b) $\frac{13!}{5!}$
(c) $\frac{13!}{6!}$
(d) $\frac{13!}{7!}$

2. You are given 5 books and 7 bookshelves. How many ways are there to place these books on the shelves? (The order on the shelves matters.)

$\frac{(n+m-1)}{(n-1)!} = \frac{(7+5-1)!}{(7-1)!} = \frac{11!}{6!}$

(a) $\binom{7}{5}$
(b) $\frac{11!}{6!}$
(c) $\frac{11!}{7!}$
(d) $\frac{12!}{7!}$

3. A password consists of 6 or 7 characters, each character being an uppercase letter or a lowercase letter. A password must contain at least one uppercase letter. How many passwords are there?

26 26

$52^6 + 52^7 - 26^6 - 26^7$

(a) $52^6 + 52^7$
(b) $26 \cdot 52^5 + 26 \cdot 52^6$
(c) $52^6 + 52^7 - 26^6 - 26^7$
(d) none of the above

4. In a group of 20 people,

- 6 are blond,
- 7 have green eyes,
- 11 are not blond and do not have green eyes.

How many people are blond and have green eyes?

$S = \{1, 2 \ldots 20\}$

$20 - 11 = 9$ -> not blonde, not green eyes

(a) 3
(b) 4
(c) 5
(d) 9

$6 + 7 - 9$

$13 - 9$

4

5. How many bitstrings of length 55 start with 101 or end with 1111?

(a) $2^{52} + 2^{51}$

(b) $2^{55} - 2^{48}$

(c) $2^{55} - 2^{52} - 2^{51}$

(d) $2^{52} + 2^{51} - 2^{48}$

$2^{55-3} + 2^{55-4} - 2^{55-3-4}$

$2^{52} + 2^{51} - 2^{48}$

6. Each person in a group of n people has a last name consisting of two uppercase letters. For what values of n can we guarantee that there are at least two people with the same last name?

(a) $n \geq 26$

(b) $n \geq 52$

(c) $n \geq 676$

(d) $n \geq 677$

7. How many bitstrings of length 13 contain exactly 3 zeros?

(a) $\binom{13}{10}$

(b) $13!/3!$

(c) $2^{13} - \binom{13}{3}$

(d) $2^{13} - 3$

$\binom{n}{k} = \binom{n}{n-k} = \binom{13}{13-3} = \binom{13}{10}$

excludes "000"

8. What is the coefficient of $x^{12}y^{12}$ in the expansion of $(3x - 7y)^{24}$?

(a) $= 3^{12}7^{12}\binom{24}{12}$

(b) $(3x)^{12}(-7y)^{12}\binom{24}{12}$

(c) $21^{12}\binom{24}{12}$

(d) $(3x)^{12}(7y)^{12}\binom{24}{12}$

$= ((3x) + (-7))^{24}$

$\sum^{24} \binom{24}{k} 3^{24-k} 7^{k}$

$\binom{24}{12} 3^{24-12} 7^{12}$

$\binom{24}{12} 3^{12} 7^{12}$

$\binom{24}{12} 21^{12}$

9. Which of the following is true?

(a) $\sum_{k=0}^{n} 5^k \binom{n}{k} = 6^n$

(b) $\sum_{k=0}^{n} 4^{n-k} 5^k \binom{n}{k} = 8^n$

(c) $\sum_{k=0}^{n} 5^k \binom{n}{k} = 5^n$

(d) $\sum_{k=0}^{n} 4^k 5^{n-k} \binom{n}{k} = 20^n$

10. How many strings can be obtained by rearranging the letters of the word

POOPERSCOOPER

(a) $13!$

(b) $\binom{13}{4}\binom{9}{3}\binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{1}{1}$

(c) $\binom{13}{4}\binom{9}{3}\binom{6}{2}\binom{4}{2}$

(d) $4!3!2!2!1!1!$

11. The function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$\begin{aligned} f(0) &= 14 \\ f(n+1) &= f(n) + 4n - 5 \text{ for } n \geq 0 \end{aligned}$$

What is $f(n)$?

(a) $f(n) = 2n^2 + 6n + 14$

(b) $f(n) = 2n^2 - 6n + 14$

(c) $f(n) = 2n^2 + 7n + 14$

(d) $f(n) = 2n^2 - 7n + 14$

12. Consider the following recursive algorithm FIB, which takes as input an integer $n \geq 0$:

Algorithm FIB(n):
if $n = 0$ or $n = 1$
then $f = n$
else $f =$ FIB$(n-1)$ + FIB$(n-2)$
endif;
return f

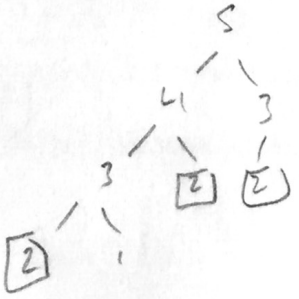
When running FIB(5), how many calls are there to FIB(2)?

(a) 1

(b) 2

(c) 3

(d) 4



13. The Fibonacci numbers are defined as follows: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.
Consider again the recursive algorithm FIB, which takes as input an integer $n \geq 0$:

Algorithm FIB(n):
if $n = 0$ or $n = 1$
then $f = n$
else $f =$ FIB$(n-1) +$ FIB$(n-2)$
endif;
return f

For $n \geq 2$, run algorithm FIB(n) and let a_n be the number of times that FIB(0) is called.

(a) For $n \geq 2$, $a_n = f_{n-1}$
(b) For $n \geq 2$, $a_n = f_n$
(c) For $n \geq 2$, $a_n = f_n - 1$
(d) For $n \geq 2$, $a_n = f_{n+1}$

14. What does the summation $\sum_{k=7}^{n} \binom{k-1}{6}$ count?

$n = k-1$
$= 7-1$
$= 6$

(a) The number of subsets of $\{1, 2, \ldots, n\}$ having size 5.
(b) The number of subsets of $\{1, 2, \ldots, n\}$ having size 6.
(c) The number of subsets of $\{1, 2, \ldots, n\}$ having size 7.
(d) The number of pints of beer you drink when running algorithm BEER(n).

15. If you flip a fair coin 4 times, what is the probability that the coin comes up head exactly twice?

(a) $1/\binom{4}{2}$
(b) $2/2^4$
(c) $2^4/\binom{4}{2}$
(d) $\binom{4}{2}/2^4$

$\frac{\binom{4}{2}}{2^4}$

HHTT
TTHH
THHT
HTTH

16. If you choose an element x uniformly at random from the set $\{1, 2, \ldots, 100\}$, what is the probability that x is divisible by 4 or 5?

(a) 9/100
(b) 1/5
(c) 2/5
(d) 45/100

A = "x ÷ by 4"
B = "x ÷ by 5"

$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

$= \frac{25}{100} + \frac{20}{100} - \frac{5}{100}$

$= \frac{40}{100}$

17. If you answer each question in this midterm by choosing an answer uniformly at random, what is the probability that you get all answers correct?

(a) $1/17^4$

(b) $1/4^{17}$

(c) $3^{17}/4^{17}$

(d) $4^{17}/3^{17}$

$\frac{1}{4^{17}}$