Midterm COMP 2804

February 27, 2014

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.
- This is a closed-book exam.
- Calculators are not allowed.

Marking scheme: Each of the 17 questions is worth 1 mark.

- Newton: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

1. On a table, you see three types of fruit: apples, bananas, and oranges. There are $m \geq 2$ apples, $n \geq 2$ bananas, and $k \geq 2$ oranges. How many ways are there to choose 7 pieces of fruit, if you must take at least two pieces of each type?

   (a) $\binom{m+n+k}{7} - (m+n+k)$
   (b) $\binom{m+n+k}{7} - \binom{m}{2} - \binom{n}{2} - \binom{k}{2}$
   (c) $\binom{m}{3}\binom{n}{2}\binom{k}{2} + \binom{m}{2}\binom{n}{3}\binom{k}{2} + \binom{m}{2}\binom{n}{2}\binom{k}{3}$
   (d) $\binom{m}{2}\binom{n}{2}\binom{k}{2}(m+n+k)$

2. Consider 9 boys and 15 girls. How many ways are there to arrange these 24 people on a line if all boys stand next to each other and all girls stand next to each other?

   (a) $\frac{24!}{9!15!}$
   (b) $\binom{24}{9}(9!)(15!)$
   (c) $(9!)(15!)$
   (d) $2(9!)(15!)$

3. Let S be a set of size 37, and let x, y, and z be three distinct elements of S. How many subsets of S are there that contain x *and* y, but do not contain z?

   (a) 2^{33}
   (b) 2^{34}
   (c) 2^{35}
   (d) $2^{37} - 2^{35} - 2^{36}$

4. Let S be a set of size 37, and let x, y, and z be three distinct elements of S. How many subsets of S are there that contain x *or* y, but do not contain z?

   (a) $2^{36} - 2^{34}$
   (b) $2^{36} - 2^{35}$
   (c) $2^{37} - 2^{34}$
   (d) $2^{37} - 2^{35}$

5. A password consists of 12 or 13 characters, each character being one of the 10 digits $0, 1, 2, \ldots, 9$. A password must contain the digit 7 at least once. How many passwords are there?

   (a) $10^{12} + 10^{13} - 9^{12} - 9^{13}$
   (b) $12^{10} + 13^{10} - 12^{9} - 13^{9}$
   (c) $10^{12} + 10^{13} - 7^{12} - 7^{13}$
   (d) $12^{10} + 13^{10} - 12^{7} - 13^{7}$

6. Let $n \geq 7$ and $k \geq 1$ be integers, let A be the set of all bitstrings of length n that contain exactly seven 0s, and let B be the set of all bitstrings of length k that contain at least one 1. Assume there exists a one-to-one function $f : A \to B$. Which of the following is true?

   (a) $2^k - 1 < \binom{n}{7}$
   (b) $2^k - 1 \geq \binom{n}{7}$
   (c) $2^k - 1 < 2^n / \binom{n}{n-7}$
   (d) $2^k - 1 \geq 2^n / \binom{n}{n-7}$

7. What is the coefficient of $x^9 y^{16}$ in the expansion of $(7x + 21y)^{25}$?

   (a) $\binom{25}{16} 7^{16} 21^{9}$
   (b) $\binom{16}{25} 7^{9} 21^{16}$
   (c) $\binom{25}{16} 7^{25} 3^{16}$
   (d) none of the above

8. How many solutions are there to the equation $x_1 + x_2 + x_3 = 17$, where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$ are integers?

   (a) $\binom{19}{16}$
   (b) $\binom{19}{17}$
   (c) $\binom{20}{16}$
   (d) $\binom{20}{17}$

9. How many strings can be obtained by rearranging the letters of the word

POOPERSCOOPER

(a) 13!

(b) $\binom{13}{4}\binom{9}{2}\binom{7}{2}\binom{5}{3}$

(c) $\binom{13}{4}\binom{9}{3}\binom{6}{2}\binom{4}{2}$

(d) $\binom{13}{1}\binom{12}{4}\binom{8}{2}\binom{6}{1}\binom{5}{2}\binom{3}{3}$

10. The function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$\begin{aligned} f(0) &= 2 \\ f(n+1) &= f(n) + 6n - 2 \text{ for } n \geq 0 \end{aligned}$$

What is $f(n)$?

(a) $f(n) = 3n^2 - 5n + 2$

(b) $f(n) = 3n^2 + 5n + 2$

(c) $f(n) = 2n^2 - 5n + 2$

(d) $f(n) = 2n^2 + 5n + 2$

11. Consider the following recursive algorithm FIB, which takes as input an integer $n \geq 0$:

Algorithm FIB(n):
if $n = 0$ or $n = 1$
then $f = n$
else $f = \text{FIB}(n-1) + \text{FIB}(n-2)$
endif;
return f

When running FIB(7), how many calls are there to FIB(3)?

(a) 4

(b) 5

(c) 6

(d) 7

12. The Fibonacci numbers are defined as follows: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.
Consider again the recursive algorithm FIB, which takes as input an integer $n \geq 0$:

Algorithm FIB(n):
if $n = 0$ or $n = 1$
then $f = n$
else $f = \text{FIB}(n-1) + \text{FIB}(n-2)$
endif;
return f

For $n \geq 3$, run algorithm FIB(n) and let a_n be the number of times that FIB(2) is called. Which of the following is true?

(a) For $n \geq 3$, $a_n = f_{n-1}$
(b) For $n \geq 3$, $a_n = f_n$
(c) For $n \geq 3$, $a_n = f_{n+1}$
(d) For $n \geq 3$, $a_n = -1 + f_n$

13. Let B_n be the number of bitstrings of length n that do not contain 111. Which of the following is true?

(a) $B_n = B_{n-1} + B_{n-2} + 2^{n-3}$
(b) $B_n = B_{n-1} + B_{n-2} + 2^{n-3} - B_{n-3}$
(c) $B_n = B_{n-1} + B_{n-2} + B_{n-3}$
(d) $B_n = B_{n-1} + B_{n-2} + B_{n-3} + 2^{n-4}$

14. A standard deck of 52 cards has 4 Kings. Consider a hand of 9 cards, chosen uniformly at random. What is the probability that there are exactly two Kings in this hand?

(a) $1 - \binom{48}{7} / \binom{52}{9}$
(b) $\{\binom{4}{2} + \binom{48}{7}\} / \binom{52}{9}$
(c) $\binom{52}{9} / \{\binom{4}{2}\binom{48}{7}\}$
(d) $\binom{4}{2}\binom{48}{7} / \binom{52}{9}$

15. We choose a bitstring of length 25 uniformly at random. What is the probability that this string contains at least two 1s?

   (a) $1 - (1/2)^{25} - 25(1/2)^{25}$
   (b) $1 + (1/2)^{25} - 25(1/2)^{25}$
   (c) $\sum_{k=2}^{25} \binom{25}{k} (1/2)^k$
   (d) none of the above

16. Consider three people, each one having a uniformly random birthday (out of 365 days; we ignore leap years). What is the probability that at least two of them have the same birthday?

   (a) $1 - \frac{365^2}{364 \cdot 363}$
   (b) $1 - \frac{364 \cdot 363}{365^2}$
   (c) $1 - \binom{3}{2}/365^3$
   (d) $1 - \{\binom{3}{2} + \binom{3}{3}\}/365^3$

17. What is Simon Pratt's favorite drink?

   (a) Herbal tea
   (b) India Pale Ale
   (c) Poutine
   (d) None of the above, because Simon doesn't like beer

Answers: 1C, 2D, 3B, 4A, 5A, 6B, 7C, 8B, 9D, 10A, 11B, 12A, 13C, 14D, 15A, 16B, 17B