COMP 2804 — Solutions Assignment 3

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: James Bond
- Student number: 007

Question 2: You flip a fair coin six times.

- What is the sample space? (Give the answer in plain English; do not list all elements of the sample space.)
- Define the events

$$A = \text{"the coin comes up heads at least four times"},$$

$$B = \text{"the number of heads is equal to the number of tails"},$$

and

$$C = \text{"there are at least four consecutive heads"}.$$

Determine $\Pr(A)$, $\Pr(B)$, $\Pr(C)$, $\Pr(A \mid B)$, and $\Pr(C \mid A)$. Show your work.

Solution: The sample space S is the set of all strings of 6 characters, where each character is H or T.

First observe that the sample space has size $2^6 = 64$, i.e.,

$$|S| = 64.$$

The event A is the subset consisting of all strings in S that contain 4, 5, or 6 H's. It follows that

$$|A| = \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 15 + 6 + 1 = 22.$$

It follows that

$$\Pr(A) = \frac{|A|}{|S|} = \frac{22}{64} = \frac{11}{32}.$$

The event B is the subset consisting of all strings in S that contain as many H's as T's, which is the same as the subset consisting of all strings in S that contain exactly 3 H's. Thus,

$$|B| = \binom{6}{3} = 20.$$

It follows that

$$\Pr(B) = \frac{|B|}{|S|} = \frac{20}{64} = \frac{5}{16}.$$

The event C consists of the following strings:

- Exactly 4 consecutive heads: H^4TH, H^4TT, TH^4T, HTH^4, and TTH^4. There are 5 strings of this type.
- Exactly 5 consecutive heads: H^5T and TH^5. There are 2 strings of this type.
- Exactly 6 consecutive heads: H^6. There is 1 string of this type.

Thus,

$$|C| = 5 + 2 + 1 = 8.$$

It follows that

$$\Pr(C) = \frac{|C|}{|S|} = \frac{8}{64} = \frac{1}{8}.$$

By the definition of conditional probability, we have

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Observe that $A \cap B = \emptyset$, because there are no strings in the sample space S that satisfy both A and B. It follows that

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(\emptyset)}{\Pr(B)} = \frac{0}{\Pr(B)} = 0.$$

By the definition of conditional probability, we have

$$\Pr(C \mid A) = \frac{\Pr(C \cap A)}{\Pr(A)}.$$

Observe that $C \cap A = C$. It follows that

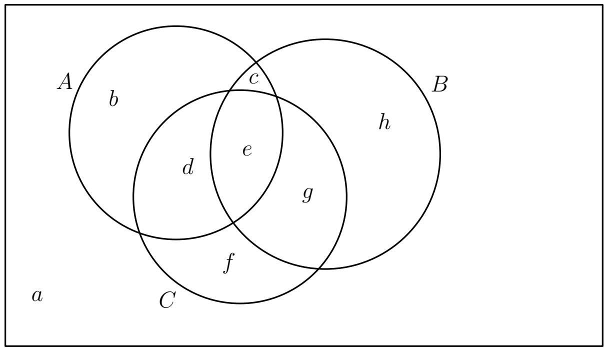
$$\Pr(C \mid A) = \frac{\Pr(C \cap A)}{\Pr(A)} = \frac{\Pr(C)}{\Pr(A)} = \frac{1/8}{11/32} = 4/11.$$

Question 3: You are given three events A, B, and C in a sample space S. Is the following true or false?

$$\Pr\left(A \cap \overline{B} \cap \overline{C}\right) = \Pr(A \cup B \cup C) - \Pr(B) - \Pr(C) + \Pr\left(B \cap C\right).$$

Justify your answer.

First Solution: We draw a Venn diagram for the events A, B, and C:



In the left-hand side, i.e., $\Pr\left(A \cap \overline{B} \cap \overline{C}\right)$, we only take care of region b. Let us see what the right-hand side does:

- $\Pr(A \cup B \cup C)$: Takes care of b, c, d, e, f, g, and h, each with a plus-sign.
- $\Pr(B)$: Takes care of c, e, g, and h, each with a minus-sign.
- $\Pr(C)$: Takes care of d, e, f, and g, each with a minus-sign.
- $\Pr(B \cap C)$: Takes care of e and g, each with a plus-sign.
- We see that the entire right-hand side only takes care of the region b.

We conclude that

$$\Pr\left(A \cap \overline{B} \cap \overline{C}\right) = \Pr(A \cup B \cup C) - \Pr(B) - \Pr(C) + \Pr(B \cap C).$$

Second Solution: We want to prove that

$$\Pr\left(A \cap \overline{B} \cap \overline{C}\right) = \Pr(A \cup B \cup C) - \Pr(B) - \Pr(C) + \Pr(B \cap C). \tag{1}$$

We start by rewriting this equation. By the inclusion-exclusion formula, we have

$$\Pr(B \cup C) = \Pr(B) + \Pr(C) - \Pr(B \cap C).$$

Using this, (1) becomes

$$\Pr\left(A \cap \overline{B} \cap \overline{C}\right) = \Pr(A \cup B \cup C) - \Pr(B \cup C),$$

which we rewrite as

$$\Pr(A \cup B \cup C) = \Pr(B \cup C) + \Pr\left(A \cap \overline{B} \cap \overline{C}\right). \tag{2}$$

By De Morgan, we have

$$\overline{B} \cap \overline{C} = \overline{B \cup C}.$$

Thus, (2) becomes

$$\Pr(A \cup B \cup C) = \Pr(B \cup C) + \Pr\left(A \cap \overline{B \cup C}\right). \tag{3}$$

Since $A \cup B \cup C$ is the disjoint union of $B \cup C$ and $A \cap \overline{B \cup C}$, (3) holds.

Question 4: Consider two events A and B in a sample space S.

- Assume that $\Pr(A) = 1/2$ and $\Pr\left(B \mid \overline{A}\right) = 3/5$. Determine $\Pr(A \cup B)$.

  Solution: First note that

  $$\Pr\left(\overline{A}\right) = 1 - \Pr(A) = 1 - 1/2 = 1/2.$$

  We have

  $$\Pr\left(B \mid \overline{A}\right) = \frac{\Pr\left(B \cap \overline{A}\right)}{\Pr\left(\overline{A}\right)} = \frac{\Pr\left(B \cap \overline{A}\right)}{1/2} = 2 \cdot \Pr\left(B \cap \overline{A}\right),$$

  implying that

  $$\Pr\left(B \cap \overline{A}\right) = \frac{\Pr\left(B \mid \overline{A}\right)}{2} = \frac{3/5}{2} = 3/10.$$

  By drawing a Venn diagram, you will see that $A \cup B$ is the disjoint union of A and $B \cap \overline{A}$. It follows that

  $$\Pr(A \cup B) = \Pr(A) + \Pr\left(B \cap \overline{A}\right) = 1/2 + 3/10 = 4/5.$$

- Assume that $\Pr(A \cup B) = 5/6$ and $\Pr\left(\overline{A} \mid \overline{B}\right) = 1/3$. Determine $\Pr(B)$.

  Solution: Using De Morgan, we have

  $$\overline{A} \cap \overline{B} = \overline{A \cup B},$$

  implying that

  $$\Pr\left(\overline{A} \cap \overline{B}\right) = \Pr\left(\overline{A \cup B}\right) = 1 - \Pr(A \cup B) = 1 - 5/6 = 1/6.$$

  Since

  $$\Pr\left(\overline{A} \mid \overline{B}\right) = \frac{\Pr\left(\overline{A} \cap \overline{B}\right)}{\Pr\left(\overline{B}\right)} = \frac{1/6}{\Pr\left(\overline{B}\right)},$$

  it follows that

  $$\Pr\left(\overline{B}\right) = \frac{1/6}{\Pr\left(\overline{A} \mid \overline{B}\right)} = \frac{1/6}{1/3} = 1/2,$$

  implying that

  $$\Pr(B) = 1 - \Pr\left(\overline{B}\right) = 1 - 1/2 = 1/2.$$

Question 5: Let S be a set consisting of 6 positive integers and 8 negative integers. Choose a 4-element subset of S uniformly at random, and multiply the elements in this subset. Denote the product by x. Determine the probability that $x > 0$. Show your work.

Solution: Since we choose a 4-element subset of a set of $6 + 8 = 14$ elements, the sample space has size $\binom{14}{4}$. There are three possibilities for the product x to be positive:

- All elements in the subset are positive: The number of such subsets is equal to $\binom{6}{4}$.
- Two elements in the subset are positive, and the other two are negative. The number of such subsets is equal to $\binom{6}{2}\binom{8}{2}$.
- All elements in the subset are negative: The number of such subsets is equal to $\binom{8}{4}$.

We conclude that

$$\Pr(x > 0) = \frac{\binom{6}{4} + \binom{6}{2}\binom{8}{2} + \binom{8}{4}}{\binom{14}{4}}.$$

Question 6: Give an example of a sample space S and six events A, B, C, D, E, and F such that

- $\Pr(A \mid B) = \Pr(A)$,
- $\Pr(C \mid D) < \Pr(C)$,
- $\Pr(E \mid F) > \Pr(E)$.

Justify your answer.
Hint: The sequence of six events may contain duplicates. Try to make the sample space S as small as you can.

Solution: The hint tells us that we should make the sample space as small as we can. Since we cannot use a sample space of size one to answer the question (because there are only two possible events in such a sample space), let us try a sample space of size two:

$$S = \{0, 1\},$$

with the uniform probability function, i.e.,

$$\Pr(0) = \Pr(1) = 1/2.$$

- We take $A = \{0\}$ and $B = S = \{0, 1\}$. Note that

  $$\Pr(B) = 1.$$

  We have

  $$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} = \frac{\Pr(A)}{1} = \Pr(A).$$

- We take $C = \{0\}$ and $D = \{1\}$. Then

  $$\Pr(C \mid D) = \frac{\Pr(C \cap D)}{\Pr(D)} = \frac{\Pr(\emptyset)}{\Pr(D)} = \frac{0}{\Pr(D)} = 0 < 1/2 = \Pr(C).$$

- We take $E = \{0\}$ and $F = \{0\}$. Then

$$\Pr(E \mid F) = \frac{\Pr(E \cap F)}{\Pr(F)} = \frac{\Pr(F)}{\Pr(F)} = 1 > 1/2 = \Pr(E).$$

Question 7: Let $n \geq 3$ be an integer, consider a uniformly random permutation of the set $\{1, 2, \ldots, n\}$, and define the events

$$A = \text{"in this permutation, 2 is to the left of 3"}$$

and

$$B = \text{"in this permutation, 1 is to the left of 2 and 1 is to the left of 3"}.$$

Are these two events independent? Justify your answer.
Hint: Use the Product Rule to count the number of permutations that define A and B.

Solution:

- We start by counting the number of permutations in which 2 is to the left of 3. Here is the procedure to obtain such a permutation:
  - Choose two positions in $\{1, 2, \ldots, n\}$.
  - Write 2 in the leftmost chosen position.
  - Write 3 in the rightmost chosen position.
  - Write the numbers $1, 4, 5, 6, \ldots, n$ in the unchosen $n - 2$ positions.

  The number of ways to do the procedure is equal to

  $$\binom{n}{2} \cdot 1 \cdot 1 \cdot (n-2)! = \frac{n!}{2}.$$

  This number is equal to the number of permutations that define the event A.

- Next we count the number of permutations in which 1 is to the left of 2 and 1 is to the left of 3. Here is the procedure to obtain such a permutation:
  - Choose three positions in $\{1, 2, \ldots, n\}$.
  - Write 1 in the leftmost chosen position.
  - Write 2 and 3 in the other two chosen positions.
  - Write the numbers $4, 5, 6, \ldots, n$ in the unchosen $n - 3$ positions.

  The number of ways to do the procedure is equal to

  $$\binom{n}{3} \cdot 1 \cdot 2 \cdot (n-3)! = \frac{n!}{3}.$$

  This number is equal to the number of permutations that define the event B.

The sample space S is the set of all permutations of $\{1, 2, \ldots, n\}$. Since the sample space S has size $n!$, we obtain

$$\Pr(A) = \frac{|A|}{|S|} = \frac{n!/2}{n!} = 1/2$$

and

$$\Pr(B) = \frac{|B|}{|S|} = \frac{n!/3}{n!} = 1/3.$$

In order to decide whether or not the events A and B are independent, we have to decide whether or not $\Pr(A \cap B)$ is equal to $\Pr(A) \cdot \Pr(B)$. Note that $A \cap B$ is the event

$$A \cap B = \text{“in this permutation, 1 is to the left of 2 and 2 is to the left of 3”}.$$

Using the Product Rule, the number of such permutations is equal to

$$\binom{n}{3} \cdot 1 \cdot 1 \cdot 1 \cdot (n-3)! = \frac{n!}{6}.$$

Therefore,

$$\Pr(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{n!/6}{n!} = 1/6.$$

We conclude that

$$\Pr(A \cap B) = 1/6 = 1/2 \cdot 1/3 = \Pr(A) \cdot \Pr(B),$$

i.e., A and B are independent.

Question 8: Consider two players P_1 and P_2:

- P_1 has one fair coin.
- P_2 has two coins. One of them is fair, whereas the other one is 2-headed (Her Majesty is on both sides of this coin).

The two players P_1 and P_2 play a game in which they alternate making turns: P_1 starts, after which it is P_2's turn, after which it is P_1's turn, after which it is P_2's turn, etc.

- When it is P_1's turn, she flips her coin once.
- When it is P_2's turn, he does the following:
  - P_2 chooses one of his two coins uniformly at random. Then he flips the chosen coin once.
  - If the first flip did not results in heads, then P_2 repeats this process one more time: P_2 again chooses one of his two coins uniformly at random and flips the chosen coin once.

The player who flips heads first is the winner of the game.

- Determine the probability that P_2 wins this game, assuming that all random choices and coin flips made are mutually independent. Justify your answer.

Solution: Since the game ends as soon as a player flips H, the sample space is

$$S = \{T^n H : n \geq 0\}.$$

- P_1 flips H with probability $1/2$ and T with probability $1/2$.
- P_2 flips H if
  - he picks the fair coin, flips it and it comes up H; this happens with probability $\frac{1}{2} \cdot \frac{1}{2} = 1/4$
  - or he picks the 2-headed coin; this happens with probability $1/2$.

  Thus, P_2 flips H with probability $1/4 + 1/2 = 3/4$; P_2 flips T with probability $1/4$.

Let A be the event that P_1 wins the game. This event corresponds to the subset

$$\{T^{3n} H : n \geq 0\}.$$

We have

$$\Pr\left(T^{3n} H\right) = \left(\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4}\right)^n \frac{1}{2} = \left(\frac{1}{32}\right)^n \frac{1}{2}.$$

It follows that

$$\begin{aligned}
\Pr(A) &= \sum_{n=0}^{\infty} \Pr\left(T^{3n} H\right) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{32}\right)^n \frac{1}{2} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{32}\right)^n .
\end{aligned}$$

Using

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

it follows that

$$\Pr(A) = \frac{1}{2} \cdot \frac{1}{1 - 1/32} = \frac{16}{31}.$$

We conclude that the probability that P_2 wins the game is equal to

$$1 - \Pr(A) = 1 - \frac{16}{31} = \frac{15}{31}.$$

Question 9: You would like to generate a *biased* random bit: With probability 2/3, this bit is 0, and with probability 1/3, it is 1. You find a *fair* coin in your pocket: This coin comes up heads (H) with probability 1/2 and tails (T) with probability 1/2. In this question, you will show that this coin can be used to generate a biased random bit.

Consider the following recursive algorithm GetBiasedBit, which does not take any input:

```
Algorithm GetBiasedBit:

    // all coin flips made are mutually independent
    flip the coin;
    if the result is H
    then return 0
    else b = GetBiasedBit
         return 1 − b
    endif
```

- The sample space S is the set of all sequences of coin flips that can occur when running algorithm GetBiasedBit. Determine this sample space S.
- Prove that algorithm GetBiasedBit returns 0 with probability 2/3.

Solution: If the coin comes up H, then the algorithm outputs a bit and terminates. Otherwise, the coin comes up T, and the algorithm repeats. In other words, the algorithm terminates (and outputs a bit) as soon as the coin comes up H for the first time. Therefore, the sample space is the set

$$S = \{T^n H : n \geq 0\}.$$

Note the following:

- If $b = 0$, then $1 - b = 1$.
- If $b = 1$, then $1 - b = 0$.

Using this, it follows from the algorithm that:

- If GetBiasedBit is called an odd number of times before terminating, the output is 0.
- If GetBiasedBit is called an even number of times before terminating, the output is 1.

Let A be the event that the algorithm outputs the bit 0. This event corresponds to the subset

$$\{T^{2n}H : n \geq 0\}.$$

It follows that

$$\begin{aligned}
\Pr(A) &= \sum_{n=0}^{\infty} \Pr\left(T^{2n}H\right) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n+1} \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n}.
\end{aligned}$$

Using

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

it follows that

$$\Pr(A) = \frac{1}{2} \cdot \frac{1}{1-1/4} = \frac{2}{3}.$$