

COMP 2804 — Solutions Assignment 2

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: James Bond
- Student number: 007

Question 2: The functions $f : \mathbb{N} \rightarrow \mathbb{N}$ and $g : \mathbb{N} \rightarrow \mathbb{N}$ are recursively defined as follows:

$$\begin{aligned}f(0) &= 1, \\f(n) &= g(n, f(n-1)) \quad \text{if } n \geq 1, \\g(m, 0) &= 0 \quad \text{if } m \geq 0, \\g(m, n) &= m + g(m, n-1) \quad \text{if } m \geq 0 \text{ and } n \geq 1.\end{aligned}$$

Solve these recurrences for f , i.e., express $f(n)$ in terms of n . Justify your answer.

Solution: After having stared at the recurrence for g for a few hours, and after having determined the values of $g(m, n)$ for some small values of n , you may guess that

$$g(m, n) = mn \text{ for all } m \geq 0 \text{ and } n \geq 0. \tag{1}$$

We will prove by induction that this guess is correct. We fix an integer $m \geq 0$.

Base case: If $n = 0$, then, by the definition of g , $g(m, n) = g(m, 0) = 0$ and $mn = m \cdot 0 = 0$.

Induction step: Let $n \geq 1$ and assume that

$$g(m, n-1) = m(n-1).$$

Then, using the recurrence for g , we get

$$\begin{aligned}g(m, n) &= m + g(m, n-1) \\&= m + m(n-1) \\&= mn.\end{aligned}$$

This proves that (1) holds.

Now that we know that g returns the product of its two arguments, let us look at f :

$$\begin{aligned}f(0) &= 1, \\f(n) &= g(n, f(n-1)) = n \cdot f(n-1) \quad \text{if } n \geq 1.\end{aligned}$$

This looks like the recurrence for the factorial function. We prove below, again by induction, that this is correct. That is, we prove that

$$f(n) = n! \text{ for all } n \geq 0. \tag{2}$$

Base case: If $n = 0$, then, by the definition of f , $f(n) = f(0) = 1$ and $n! = 0! = 1$.

Induction step: Let $n \geq 1$ and assume that

$$f(n-1) = (n-1)!.$$

Then, using the recurrence for f , we get

$$\begin{aligned} f(n) &= g(n, f(n-1)) \\ &= n \cdot f(n-1) \\ &= n \cdot (n-1)! \\ &= n!. \end{aligned}$$

This proves that (2) holds.

Question 3: The function $f : \mathbb{N} \rightarrow \mathbb{Z}$ is defined by

$$f(n) = 2n(n-6)$$

for each integer $n \geq 0$. Derive a recursive form of this function f . Show your work.

Solution: Our goal is to define the function f in the following way:

- Base case: Say what $f(0)$ is.
- Recurrence: Come up with an equation of the form $f(n) = f(n-1)$ plus some expression in n .

The first part is easy: If the solution to the recurrence must be $f(n) = 2n(n-6)$, then $f(0)$ must be 0.

How to get the recurrence: We want an equation of the form $f(n)$ is equal to $f(n-1)$ plus some term. This term is equal to $f(n) - f(n-1)$. Since we know that the solution to the recurrence must be $f(n) = 2n(n-6)$, we get

$$\begin{aligned} f(n) - f(n-1) &= 2n(n-6) - 2(n-1)((n-1)-6) \\ &= 2n(n-6) - 2(n-1)(n-7) \\ &= (2n^2 - 12n) - 2(n^2 - 8n + 7) \\ &= 4n - 14. \end{aligned}$$

Thus, the recursive form for the function f is given by

$$\begin{aligned} f(0) &= 0, \\ f(n) &= f(n-1) + 4n - 14 \text{ if } n \geq 1. \end{aligned}$$

Question 4: The function $f : \mathbb{N}^3 \rightarrow \mathbb{N}$ is defined as follows:

$$f(k, n, 0) = k + n \text{ if } k \geq 0 \text{ and } n \geq 0, \quad (3)$$

$$f(k, 0, 1) = 0 \text{ if } k \geq 0, \quad (4)$$

$$f(k, 0, 2) = 1 \text{ if } k \geq 0, \quad (5)$$

$$f(k, 0, i) = k \text{ if } k \geq 0 \text{ and } i \geq 3, \quad (6)$$

$$f(k, n, i) = f(k, f(k, n - 1, i), i - 1) \text{ if } k \geq 0, i \geq 1 \text{ and } n \geq 1. \quad (7)$$

Determine $f(2, 3, 2)$. Show your work.

Solution: According to (7), we have

$$f(2, 3, 2) = f(2, f(2, 2, 2), 1). \quad (8)$$

We first compute $f(2, 2, 2)$: According to (7), we have

$$f(2, 2, 2) = f(2, f(2, 1, 2), 1). \quad (9)$$

Thus, we have to compute $f(2, 1, 2)$:

$$\begin{aligned} f(2, 1, 2) &= f(2, f(2, 0, 2), 1) \quad (\text{by (7)}) \\ &= f(2, 1, 1) \quad (\text{by (5)}) \\ &= f(2, f(2, 0, 1), 0) \quad (\text{by (7)}) \\ &= f(2, 0, 0) \quad (\text{by (4)}) \\ &= 2 \quad (\text{by (3)}) \end{aligned}$$

Substitute $f(2, 1, 2) = 2$ into (9):

$$\begin{aligned} f(2, 2, 2) &= f(2, f(2, 1, 2), 1) \\ &= f(2, 2, 1) \\ &= f(2, f(2, 1, 1), 0) \quad (\text{by (7)}) \end{aligned}$$

We have already seen that $f(2, 1, 1) = f(2, 1, 2) = 2$. Thus we get

$$\begin{aligned} f(2, 2, 2) &= f(2, f(2, 1, 1), 0) \\ &= f(2, 2, 0) \\ &= 4 \quad (\text{by (3)}) \end{aligned}$$

Substitute $f(2, 2, 2) = 4$ into (8):

$$\begin{aligned} f(2, 3, 2) &= f(2, f(2, 2, 2), 1) \\ &= f(2, 4, 1) \\ &= f(2, f(2, 3, 1), 0) \quad (\text{by (7)}) \end{aligned} \quad (10)$$

We need $f(2, 3, 1)$: According to (7), we have

$$f(2, 3, 1) = f(2, f(2, 2, 1), 0).$$

We have already seen that $f(2, 2, 1) = f(2, 2, 2) = 4$. This gives

$$\begin{aligned} f(2, 3, 1) &= f(2, f(2, 2, 1), 0) \\ &= f(2, 4, 0) \\ &= 6 \quad (\text{by (3)}) \end{aligned}$$

Substitute $f(2, 3, 1) = 6$ into (10):

$$\begin{aligned} f(2, 3, 2) &= f(2, f(2, 3, 1), 0) \\ &= f(2, 6, 0) \\ &= 8 \quad (\text{by (3)}) \end{aligned}$$

Question 5: A *maximal run of ones* in a bitstring is a maximal consecutive substring of ones. For example, the bitstring 1100011110100111 has four maximal runs of ones: 11, 1111, 1, and 111. These maximal runs have lengths 2, 4, 1, and 3, respectively.

Let $n \geq 1$ be an integer and let B_n be the number of bitstrings of length n that do not contain any maximal run of ones of odd length; in other words, every maximal run of ones in these bitstrings has an even length.

- Determine B_1 , B_2 , and B_3 .
- Determine the value of B_n , i.e., express B_n in terms of numbers that we have seen in class. Justify your answer. *Hint:* Derive a recurrence.

Solution:

- We determine B_1 . There are 2 bitstrings of length 1:
 - 0: does not have any run of ones.
 - 1: this is a maximal run of ones having odd length.

Thus,

$$B_1 = 1.$$

- We determine B_2 . There are 4 bitstrings of length 2:
 - 00: does not have any run of ones.
 - 01: contains a maximal run of ones having odd length.
 - 10: contains a maximal run of ones having odd length.

- 11: this is a maximal run of ones having even length.

Thus,

$$B_2 = 2.$$

- We determine B_3 . There are 8 bitstrings of length 3:
 - 000: does not have any run of ones.
 - 001: contains a maximal run of ones having odd length.
 - 010: contains a maximal run of ones having odd length.
 - 100: contains a maximal run of ones having odd length.
 - 011: contains a maximal run of ones having even length.
 - 101: contains two maximal runs of ones having odd length.
 - 110: contains a maximal run of ones having even length.
 - 111: this is a maximal run of ones having odd length.

Thus,

$$B_3 = 3.$$

We derive a recurrence for B_n . Let $n \geq 3$ and consider all bitstrings of length n that do not contain any maximal run of ones of odd length. The number of such bitstrings is equal to B_n . We divide these bitstrings into two groups:

- Those that start with 0. The rest of these strings have length $n - 1$ and do not contain any maximal run of ones of odd length. Thus, the number of strings in this group is equal to B_{n-1} .
- Those that start with 1. In these strings, the second bit must be equal to 1, because otherwise, the first bit would be a maximal run of ones having an odd length. The rest of these strings have length $n - 2$ and do not contain any maximal run of ones of odd length. Thus, the number of strings in this group is equal to B_{n-2} .

We conclude that

$$B_n = B_{n-1} + B_{n-2},$$

which is the Fibonacci recurrence. Recall that the Fibonacci numbers are given by

$$f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, \dots$$

and

$$B_1 = 1, B_2 = 2, B_3 = 3, B_4 = 5, B_5 = 8, B_6 = 13, \dots$$

From this, we get

$$B_n = f_{n+1} \text{ if } n \geq 1.$$

Question 6: Let $n \geq 1$ be an integer and let S_n be the number of ways in which n can be written as a sum of 1s and 2s; the order in which the 1s and 2s occur in the sum matters. For example, $S_3 = 3$, because

$$3 = 1 + 1 + 1 = 1 + 2 = 2 + 1.$$

- Determine S_1 , S_2 , and S_4 .
- Determine the value of S_n , i.e., express S_n in terms of numbers that we have seen in class. Justify your answer. *Hint:* Derive a recurrence.

Solution:

- We determine S_1 . The integer 1 can be written in only one way as a sum of 1s and 2s:

$$1 = 1.$$

Thus,

$$S_1 = 1.$$

- We determine S_2 . The integer 2 can be written in two ways as a sum of 1s and 2s:

$$2 = 1 + 1$$

and

$$2 = 2.$$

Thus,

$$S_2 = 2.$$

- We determine S_4 . The integer 4 can be written in five ways as a sum of 1s and 2s:

$$4 = 1 + 1 + 1 + 1,$$

$$4 = 1 + 1 + 2,$$

$$4 = 1 + 2 + 1,$$

$$4 = 2 + 1 + 1,$$

and

$$4 = 2 + 2.$$

Thus,

$$S_4 = 5.$$

We derive a recurrence for S_n . Let $n \geq 3$ and consider all ways to write n as a sum of 1s and 2s. The number of such sums is equal to S_n . We divide these sums into two groups:

- Those that start with 1. The rest of these sums are sums of 1s and 2s that add up to $n - 1$. Thus, the number of sums in this group is equal to S_{n-1} .
- Those that start with 2. The rest of these sums are sums of 1s and 2s that add up to $n - 2$. Thus, the number of sums in this group is equal to S_{n-2} .

We conclude that

$$S_n = S_{n-1} + S_{n-2},$$

which is the Fibonacci recurrence. Recall that the Fibonacci numbers are given by

$$f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, \dots$$

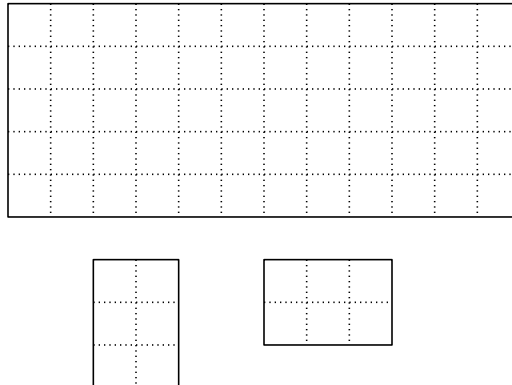
and

$$S_1 = 1, S_2 = 2, S_3 = 3, S_4 = 5, S_5 = 8, S_6 = 13, \dots$$

From this, we get

$$S_n = f_{n+1} \text{ if } n \geq 1.$$

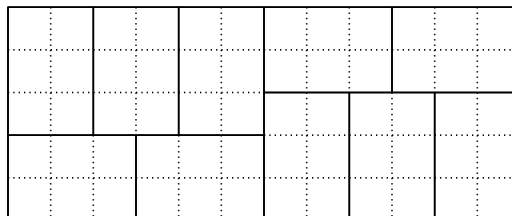
Question 7: Let n be a positive integer and consider a $5 \times n$ board B_n consisting of $5n$ cells, each one having sides of length one. The top part of the figure below shows B_{12} .



A *brick* is a horizontal or vertical board consisting of $2 \times 3 = 6$ cells; see the bottom part of the figure above. A *tiling* of the board B_n is a placement of bricks on the board such that

- the bricks exactly cover B_n and
- no two bricks overlap.

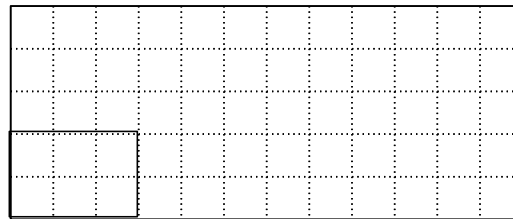
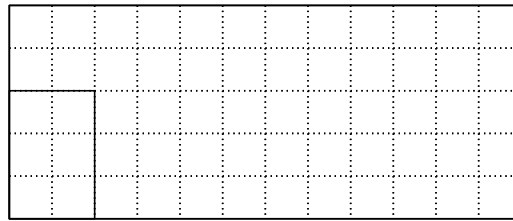
The figure below shows a tiling of B_{12} .



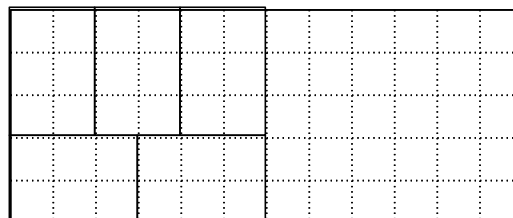
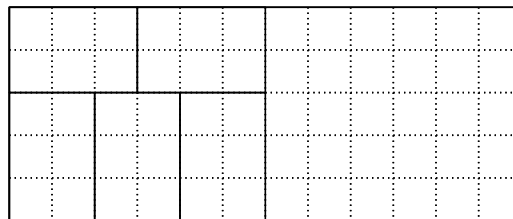
Let T_n be the number of different tilings of the board B_n .

- Let $n \geq 6$ be a multiple of 6. Determine the value of T_n . Justify your answer. *Hint:* Derive a recurrence.
- Let n be a positive integer that is not a multiple of 6. Prove that $T_n = 0$.

Solution: In any tiling of the board B_n , there are two possibilities for the bottom-left brick, as indicated in the following figure:



The top tiling in the above figure can only be continued as in the top figure below; the bottom tiling can only be continued as in the bottom figure below:



Consider all possible tilings of B_n ; there are T_n many of these. Divide them into two groups, depending on whether the bottom-left brick is vertical (i.e., a 3×2 brick) or horizontal (i.e., a 2×3 brick). In each group, the part of the tiling within the leftmost 5×6 part of

the board B_n is fixed. The remaining part of the board is B_{n-6} , and there are T_{n-6} many ways to tile it. We conclude that

$$\begin{aligned} T_6 &= 2, \\ T_n &= 2 \cdot T_{n-6} \text{ if } n \geq 12 \text{ and } n \text{ is a multiple of } 6. \end{aligned}$$

We will prove that, for $n \geq 6$ a multiple of 6,

$$T_n = 2^{n/6}.$$

Base case: If $n = 6$, then $T_6 = 2$ and $2^{n/6} = 2^1 = 2$.

Induction step: Let $n \geq 12$ be a multiple of 6 and assume that

$$T_{n-6} = 2^{(n-6)/6}.$$

Then, using the recurrence for T , we get

$$\begin{aligned} T_n &= 2 \cdot T_{n-6} \\ &= 2 \cdot 2^{(n-6)/6} \\ &= 2^{1+(n-6)/6} \\ &= 2^{n/6}. \end{aligned}$$

The arguments presented above imply that the board B_n can be tiled only in groups of subboards B_6 . Therefore, if the board B_n can be tiled, then B_n must contain non-overlapping copies of B_6 . This is possible only if n is a multiple of 6. In other words, if n is not a multiple of 6, then $T_n = 0$.

Question 8: Consider the following recursive algorithm SILLY, which takes as input an integer $n \geq 1$ which is a power of 2:

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Algorithm SILLY( $n$ ):
    if  $n = 1$ 
    then drink one pint of beer
    else if  $n = 2$ 
        then fart once
        else fart once;
            SILLY( $n/2$ );
            fart once
        endif
    endif

```

For n a power of 2, let $F(n)$ be the number of times you fart when running algorithm SILLY(n).

- Determine the value of $F(n)$ and prove that your answer is correct. *Hint:* Derive a recurrence.

Solution: It follows from the pseudocode that

$$F(1) = 0$$

and

$$F(2) = 1.$$

If n is a power of 2 and $n \geq 4$, then it follows from the pseudocode that

$$F(n) = 2 + F(n/2).$$

Thus, in each “step”, you fart twice and the argument is divided by 2. This suggests that $F(n)$ must be equal to $2 \log n$ plus some constant; this constant is determined by the base case $F(2) = 1$. After some trying, you will guess that

$$F(n) = 2 \log n - 1 \text{ if } n \geq 2 \text{ and } n \text{ is a power of } 2. \quad (11)$$

We prove by induction that this is correct:

Base case: If $n = 2$, then $F(n) = F(2) = 1$ and

$$2 \log n - 1 = 2 \log 2 - 1 = 2 \cdot 1 - 1 = 1.$$

Induction step: Let $n \geq 4$ be a power of 2 and assume that

$$F(n/2) = 2 \log(n/2) - 1.$$

Then, using the recurrence for F , we get

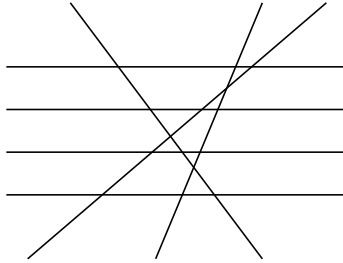
$$\begin{aligned} F(n) &= 2 + F(n/2) \\ &= 2 + (2 \log(n/2) - 1) \\ &= 2 + (2 \log n - 2 \log 2 - 1) \\ &= 2 + (2 \log n - 2 \cdot 1 - 1) \\ &= 2 \log n - 1. \end{aligned}$$

This proves that (11) holds.

Question 9: Let $m \geq 1$ and $n \geq 1$ be integers. Consider m horizontal lines and n non-horizontal lines such that

- no two of the non-horizontal lines are parallel,
- no three of the $m + n$ lines intersect in one single point.

These lines divide the plane into regions (some of which are bounded and some of which are unbounded). Denote the number of these regions by $R_{m,n}$. From the figure below, you can see that $R_{4,3} = 23$.



- Derive a recurrence for the numbers $R_{m,n}$ and use it to prove that

$$R_{m,n} = 1 + m(n + 1) + \binom{n + 1}{2}.$$

Solution: We fix a value of $m \geq 1$ and derive a recurrence for $R_{m,n}$. The base case is when $n = 0$. In this case, we have m horizontal lines, which obviously divide the plane into $m + 1$ regions. Thus,

$$R_{m,0} = m + 1.$$

Let $n \geq 1$ and consider n non-horizontal lines L_1, L_2, \dots, L_n .

- We start by adding the lines L_1, L_2, \dots, L_{n-1} to the m horizontal lines H_1, H_2, \dots, H_m . At this moment, the number of regions is $R_{m,n-1}$.
- Now we add the line L_n . This line intersects
 - each of the horizontal lines H_1, H_2, \dots, H_m and
 - each of the lines L_1, L_2, \dots, L_{n-1} .
 - Thus, there are $m + n - 1$ intersections between L_n and the lines we have already drawn. It follows that $R(m, n)$ goes through $m + n$ regions (one more than the number of intersections), and each such region is cut into two. It follows that, when adding L_n , the number of regions increases by $m + n$.

Thus, we obtain the recurrence

$$R_{m,n} = R_{m,n-1} + m + n$$

It remains to prove that

$$R_{m,n} = 1 + m(n + 1) + \binom{n + 1}{2}.$$

We do this by induction. If $n = 0$, then $R_{m,n} = R_{m,0} = m + 1$ and

$$1 + m(0 + 1) + \binom{0 + 1}{2} = 1 + m + 0 = m + 1$$

as well, proving the base case.

Let $n \geq 1$, and assume the claim is true for $n - 1$, i.e., assume that

$$R_{m,n-1} = 1 + mn + \binom{n}{2}.$$

We have to show that

$$R_{m,n} = 1 + m(n + 1) + \binom{n + 1}{2}.$$

This follows by applying the recurrence and the assumption:

$$\begin{aligned} R_{m,n} &= R_{m,n-1} + m + n \\ &= \left(1 + mn + \binom{n}{2}\right) + (m + n) \\ &= (1 + m(n + 1)) + \left(\binom{n}{2} + n\right). \end{aligned}$$

Since

$$\begin{aligned} \binom{n}{2} + n &= \frac{n(n - 1)}{2} + n \\ &= \frac{n(n - 1) + 2n}{2} \\ &= \frac{n(n + 1)}{2} \\ &= \binom{n + 1}{2}, \end{aligned}$$

$$R_{m,n} = 1 + m(n + 1) + \binom{n + 1}{2}.$$