COMP 2804 — Solutions Assignment 1

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: James Bond
- Student number: 007

Question 2: The Carleton Computer Science Society has an Academic Events Committee (AEC) consisting of five students and a Beer Committee (BC) consisting of six students (whose responsibility is to buy beer for the AEC).

- Assume there are $n \geq 6$ students in Carleton's Computer Science program. Also, assume that a student can be both on the AEC and on the BC. What is the total number of ways in which these two committees can be chosen? Justify your answer.

  Solution: To specify the two committees, there are two tasks:

  - Choose the AEC. There are $\binom{n}{5}$ ways to do this.
  - Choose the BC. There are $\binom{n}{6}$ ways to do this.

  By the Product Rule, the total number of ways to choose the two committees is equal to $\binom{n}{5}\binom{n}{6}$.

- Assume there are $n \geq 11$ students in Carleton's Computer Science program. Also, assume that a student cannot be both on the AEC and on the BC. What is the total number of ways in which these two committees can be chosen? Justify your answer.

  Solution: To specify the two committees, there are two tasks:

  - Choose the AEC. There are $\binom{n}{5}$ ways to do this.
  - Choose the BC. The five students that we have chosen for the AEC cannot be on the BC. Therefore, the number of ways to choose the BC is equal to $\binom{n-5}{6}$.

  By the Product Rule, the total number of ways to choose the two committees is equal to $\binom{n}{5}\binom{n-5}{6}$.

Question 3: Let $n \geq 1$ be an integer. Consider a tennis tournament with $2n$ participants. In the first round of this tournament, n games will be played and, thus, the $2n$ people have to be divided into n pairs. What is the number of ways in which this can be done? Justify your answer.

Solution: Denote the participants by $P_1, P_2, \ldots, P_{2n}$. To divide them into pairs, we have to do the following tasks:

- Match P_1 with one of the other $2n-1$ participants. There are $2n-1$ ways to do this.
- At this moment, 2 participants have been matched. Take a participant that has not been matched yet, for example the one whose index is minimum. Now match this guy with one of the remaining $2n-3$ participants. There are $2n-3$ ways to do this.
- At this moment, 4 participants have been matched. Take a participant that has not been matched yet, for example the one whose index is minimum. Now match this guy with one of the remaining $2n-5$ participants. There are $2n-5$ ways to do this.
- At this moment, 6 participants have been matched. Take a participant that has not been matched yet, for example the one whose index is minimum. Now match this guy with one of the remaining $2n-7$ participants. There are $2n-7$ ways to do this.
- Etc., etc.
- In the last step, $2n-2$ participants have been matched. Take a participant that has not been matched yet, for example the one whose index is minimum. Now match this guy with the only remaining participant. There is 1 way to do this.

By the Product Rule, the total number of ways to divide the $2n$ participants into n pairs is equal to

$$(2n-1)(2n-3)(2n-5)(2n-7)\cdots 5\cdot 3\cdot 1.$$

This is called the *double factorial* $(2n-1)!!$; it is the same as

$$\frac{(2n)!}{2^n\cdot n!}.$$

Question 4: The Beer Committee of the Carleton Computer Science Society has bought large quantities of 10 different types of beer. In order to test which beer students prefer, the committee does the following experiment:

- Out of the $n \geq 10$ students in Carleton's Computer Science program, 10 students are chosen.
- Each of the 10 students chosen drinks one of the 10 beers; no two students drink the same beer.

What is the number of ways in which this experiment can be done? Justify your answer.

First Solution: To do the experiment, we do the following two tasks:

- Choose 10 students. There are $\binom{n}{10}$ ways to do this.
- Divide the 10 beers among the 10 chosen students. There are $10!$ ways to do this.

By the Product Rule, the total number of ways to do the experiments is equal to

$$\binom{n}{10} \cdot 10! = \frac{n!}{(n-10)!}.$$

Second Solution: We write the beers as $B_1, B_2, \ldots, B_{10}$. To do the experiment, we do the following task:

- Take an ordered sequence of 10 students in which no student occurs more than once.

The first student in the sequence gets beer B_1, the second student gets beer B_2, the third student gets beer B_3, etc.

We have seen in class (see equation (3.1) on page 37) that the number of ways to do this is equal to

$$\frac{n!}{(n-10)!}.$$

Question 5: Consider permutations of the 26 lowercase letters a, b, c, …, z.

- How many such permutations contain the string *wine*? Justify your answer.
- How many such permutations do not contain any of the strings *wine*, *vodka*, or *coke*? Justify your answer.

Solution: Define the following sets:

- W is the set of all permutations of a, b, c, …, z that contain *wine*.
- V is the set of all permutations of a, b, c, …, z that contain *vodka*.
- C is the set of all permutations of a, b, c, …, z that contain *coke*.

The first part of the question asks for the size of the set W. Thus, we are looking for the number of permutations of the 26 letters in which the letters w, i, n, and e are next to each other, in this order. Imagine these 4 letters to be one symbol. Then we have a new alphabet consisting of

$$26 - 4 + 1 = 23$$

symbols. The size of the set W is equal to the number of permutations of these 23 symbols. Therefore,

$$|W| = 23!.$$

The second part asks for the size of the set

$$\overline{W \cup V \cup C}.$$

Define U to be the set of all permutations of a, b, c, $\ldots$, z. By the Complement Rule, we have

$$|\overline{W \cup V \cup C}| = |U| - |W \cup V \cup C|.$$

We are going to determine the two sizes on the right-hand side.

It is clear that

$$|U| = 26!.$$

For $|W \cup V \cup C|$, we are going to use the inclusion-exclusion formula:

- We have seen above that $|W| = 23!$.
- By the same reasoning, we get $|V| = 22!$ and $|C| = 23!$.
- To determine $|W \cap V|$: Replace the four letters of the word *wine* by one symbol, and replace the five letters of the word *vodka* by one symbol. Then we obtain a new alphabet with $$26 - 4 - 5 + 2 = 19$$ symbols. Thus, $|W \cap V| = 19!$.
- To determine $|W \cap C|$: Since both words *wine* and *coke* contain the letter e, $|W \cap C| = 0$.
- To determine $|V \cap C|$: Since both words *vodka* and *coke* contain the letter o, $|V \cap C| = 0$.
- To determine $|W \cap V \cap C|$: Since both words *vodka* and *coke* contain the letter o, $|W \cap V \cap C| = 0$.

We get

$$\begin{aligned} |W \cup V \cup C| &= |W| + |V| + |C| - |W \cap V| - |W \cap C| - |V \cap C| + |W \cap V \cap C| \\ &= 23! + 22! + 23! - 19! - 0 - 0 + 0 \\ &= 23! + 22! + 23! - 19!. \end{aligned}$$

We conclude that

$$|\overline{W \cup V \cup C}| = |U| - |W \cup V \cup C| = 26! - 23! - 22! - 23! + 19!.$$

Question 6: Determine the coefficient of $x^{12}y^{25}$ in the expansion of $(7x - 17y)^{37}$. Show your work.

Solution: According to Newton,

$$\begin{aligned} (7x - 17y)^{37} &= ((7x) + (-17y))^{37} \\ &= \sum_{k=0}^{37} \binom{37}{k} (7x)^{37-k}(-17y)^k. \end{aligned}$$

Take the term with $k = 25$. Then we see that the coefficient of $x^{12}y^{25}$ is equal to

$$\binom{37}{25} 7^{37-25}(-17)^{25} = -\binom{37}{25} 7^{12} 17^{25}.$$

Question 7: Let $n \geq 0$ and $k \geq 0$ be integers.

- How many bitstrings of length $n + 1$ have exactly $k + 1$ many 1s?

  Solution: We have seen in class that the answer is $\binom{n+1}{k+1}$.

- Let i be an integer with $k \leq i \leq n$. What is the number of bitstrings of length $n + 1$ that have exactly $k + 1$ many 1s and in which the rightmost 1 is at position $i + 1$?

  Solution: Any such bitstring has the following form:

  - At the first i positions, there are exactly k many 1s.
  - At position $i + 1$, there is a 1.
  - At the last $n - i$ positions, there are only 0s.

  The number of such bitstrings is equal to the number of bitstrings of length i with exactly k many 1s. The answer is thus $\binom{i}{k}$.

- Use the above two results to prove that

  $$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

  Solution: Consider all bitstrings of length $n+1$ with exactly $k+1$ many 1s. We know from the first part that there are

  $$\binom{n+1}{k+1} \tag{1}$$

  many such bitstrings.

  Divide these bitstrings into groups, based on the position of the rightmost 1. This rightmost 1 can be in any of the positions $k + 1, k + 2, k + 3, \ldots, n + 1$. We know from the second part that group i (i.e., the group in which the rightmost 1 is at position $i + 1$) contains $\binom{i}{k}$ bitstrings. Thus, the total size of all groups and, therefore, the number of bitstrings of length $n + 1$ with exactly $k + 1$ many 1s, is equal to

  $$\sum_{i=k}^{n} \binom{i}{k}. \tag{2}$$

  Since the quantities in (1) and (2) count the same things, they must be equal.

Question 8: Let $n \geq 1$ be an integer. Prove that

$$\sum_{k=1}^{n} k \binom{n}{k} = n \cdot 2^{n-1}.$$

Hint: Take the derivative of $(1+x)^n$.

Solution: According to Newton,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Take the derivative on both sides:

$$\begin{aligned} n(1+x)^{n-1} &= \sum_{k=0}^{n} k \binom{n}{k} x^{k-1} \\ &= \sum_{k=1}^{n} k \binom{n}{k} x^{k-1}. \end{aligned}$$

Take $x = 1$:

$$n \cdot 2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}.$$

Question 9: Let $n \geq 1$ be an integer and consider the set $S = \{1, 2, \ldots, 2n\}$. Let T be an arbitrary subset of S having size $n+1$. Prove that this subset T contains two elements whose sum is equal to $2n+1$.
Hint: Consider the pairs $(1, 2n)$, $(2, 2n-1)$, $(3, 2n-2)$, ..., $(n, n+1)$ and use the Pigeonhole Principle.

Solution: Consider each of the n pairs $(1, 2n)$, $(2, 2n-1)$, $(3, 2n-2)$, ..., $(n, n+1)$ to be a box. Put each of the elements of T in its box. For example, if 3 is an element of T, then we put it in the box $(3, 2n-2)$. As another example, if $2n-1$ is an element of T, then we put it in the box $(2, 2n-1)$.

By doing this, we have put $n+1$ elements in n boxes. By the Pigeonhole Principle, there is a box with at least two elements. These two elements add up to $2n+1$.