COMP 2804 — Solutions Assignment 1

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: James Bond
- Student number: 007

Question 2: The Carleton Computer Science Society has a Board of Directors consisting of one president, one vice-president, one secretary, one treasurer, and a three-person party committee (whose main responsibility is to buy beer for the other four board members). The entire board consists of seven distinct students. If there are $n \geq 7$ students in Carleton's Computer Science program, how many ways are there to choose a Board of Directors? Justify your answer.

Solution: We are going to apply the Product Rule:

- Task 1: Choose a president. There are n ways to do this.
- Task 2: Choose a vice-president. There are $n-1$ ways to do this.
- Task 3: Choose a secretary. There are $n-2$ ways to do this.
- Task 4: Choose a treasurer. There are $n-3$ ways to do this.
- Task 5: Choose a three-person party committee. There are $\binom{n-4}{3}$ ways to do this.

By the Product Rule, the number of ways to choose a Board of Directors is equal to

$$n(n-1)(n-2)(n-3)\binom{n-4}{3}.$$

Question 3: Let A be a set of size m, let B be a set of size n, and assume that $n \geq m \geq 1$. How many functions $f : A \to B$ are there that are *not* one-to-one? Justify your answer.

Solution: In class, we have seen that

- there are n^m many functions $f : A \to B$,
- there are $\frac{n!}{(n-m)!}$ many one-to-one functions $f : A \to B$.

Therefore, the number of functions $f : A \to B$ that are not one-to-one is equal to

$$n^m - \frac{n!}{(n-m)!}.$$

Question 4: In a group of 20 people,

- 6 are blond,
- 7 have green eyes,
- 11 are not blond and do not have green eyes.

How many people are blond and have green eyes? Justify your answer.

Solution: Let U be the set of people in the group, let B be the set of blonds in U, and let G be the set of people in U with green eyes. The question asks for the size of the set $B \cap G$.

We are given that

$$|U| = 20, |B| = 6, |G| = 7, |U \setminus (B \cup G)| = 11.$$

By the Complement Rule,

$$|B \cup G| = |U| - |U \setminus (B \cup G)| = 20 - 11 = 9.$$

By the Principle of Inclusion and Exclusion,

$$|B \cup G| = |B| + |G| - |B \cap G|,$$

i.e.,

$$9 = 6 + 7 - |B \cap G|.$$

It follows that

$$|B \cap G| = 4.$$

Question 5: Let $n \geq 1$ be an integer. Use the Pigeonhole Principle to prove that in any set of $n+1$ integers from $\{1, 2, \ldots, 2n\}$, there are two integers that are consecutive (i.e., differ by one).

Solution: For $i = 1, 2, \ldots, n$, define the "box" B_i to be the set $\{2i-1, 2i\}$. Thus, $B_1 = \{1, 2\}$, $B_2 = \{3, 4\}$, $B_3 = \{5, 6\}$, ..., $B_n = \{2n-1, 2n\}$.

Let S be a set of $n+1$ integers from $\{1, 2, \ldots, 2n\}$. Note that element x of S is equal to one of the two elements in the set $B_{\lceil x/2 \rceil}$. We "throw" element x in the "box" $B_{\lceil x/2 \rceil}$. Since we throw $n+1$ many elements in n boxes, it follows from the Pigeonhole Principle that there is a box with at least two elements. That means that there are two elements in S that are consecutive integers.

Question 6: Let $n \geq 1$ be an integer and consider n boys and n girls. For each of the following three cases, determine how many ways there are to arrange these $2n$ people on a straight line:

- All boys stand next to each other and all girls stand next to each other.

  Solution: Either all boys are to the left of all girls, or all boys are to the right of all girls. In either of these two cases, there are $n!$ ways to arrange the boys and $n!$ ways to arrange the girls. Therefore, the answer is

$$2(n!)^2.$$

- All girls stand next to each other.

  Solution: Number the positions $1, 2, \ldots, 2n$. If all girls stand next to each other, then there is an index i with $1 \leq i \leq n+1$, such that the girls are at the positions $i, i+1, \ldots, i+n-1$. For each of the $n+1$ possible values for i, there are $n!$ ways to arrange the girls and $n!$ ways to arrange the boys. Therefore, the answer is

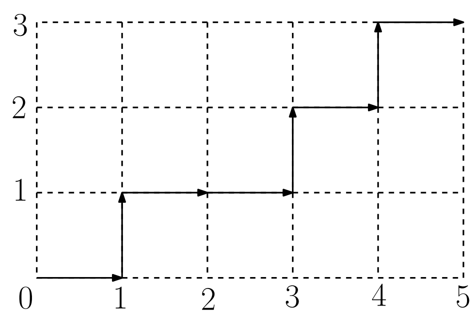
$$(n+1)(n!)^2.$$

- Boys and girls alternate.

  Solution: Either all boys are at the odd positions and all girls are at the even positions, or all boys are at the even positions and all girls are at the odd positions. In either of these two cases, there are $n!$ ways to arrange the boys and $n!$ ways to arrange the girls. Therefore, the answer is

$$2(n!)^2.$$

Question 7: Let $m \geq 1$ and $n \geq 1$ be integers. Consider a rectangle whose horizontal side has length m and whose vertical side has length n. A path from the bottom-left corner to the top-right corner is called *valid*, if in each step, it either goes one unit to the right or one unit upwards. In the example below, you see a valid path for the case when $m = 5$ and $n = 3$.



How many valid paths are there? Justify your answer.

Solution: Any valid path makes exactly m unit-steps to the right and exactly n unit-steps upwards. We encode a valid path as a strings of characters, each character being an R (for "go one unit to the right") or a U (for "go one unit upwards"). For example, the path in the figure is encoded as $RURRURUR$. A valid path is thus encoded as a string of length $m+n$, and it contains exactly m many Rs (and, therefore, exactly n many Us).

It is clear that this gives a bijection between the set of all valid paths and the set of all strings of length $m+n$ that contain m many Rs and n many Us.

By the Bijection Rule, the number of valid paths is then equal to the number of such strings, which is

$$\binom{m+n}{m}$$

which is the same as

$$\binom{m+n}{n}.$$

Question 8: Let n and k be integers with $n \geq k \geq 1$. How many solutions are there to the equation

$$x_1 + x_2 + \cdots + x_k = n,$$

where $x_1 \geq 1$, $x_2 \geq 1$, ..., $x_k \geq 1$ are integers? Justify your answer.

Hint: In class, we have seen the answer if $x_1 \geq 0$, $x_2 \geq 0$, ..., $x_k \geq 0$. Use this result.

Solution: Let A be the set of all solutions to the equation

$$x_1 + x_2 + \cdots + x_k = n,$$

where $x_1 \geq 1$, $x_2 \geq 1$, ..., $x_k \geq 1$ are integers.

Let B be the set of all solutions to the equation

$$y_1 + y_2 + \cdots + y_k = n - k,$$

where $y_1 \geq 0$, $y_2 \geq 0$, ..., $y_k \geq 0$ are integers. We have seen in class that $|B| = \binom{n-1}{k-1}$.

If we can show that there is a bijection $f : A \to B$, then, by the Bijection Rule, A has the same size as B and, therefore, the answer to this question is $|A| = \binom{n-1}{k-1}$.

Consider the following function $f : A \to B$: For any $(x_1, x_2, \ldots, x_k) \in A$,

$$f(x_1, x_2, \ldots, x_k) = (x_1 - 1, x_2 - 1, \ldots, x_k - 1).$$

Since $(x_1, x_2, \ldots, x_k) \in A$, we have

$$x_1 + x_2 + \cdots + x_k = n,$$

which is equivalent to

$$(x_1 - 1) + (x_2 - 1) + \cdots + (x_k - 1) = n - k.$$

Also observe that, since each $x_i \geq 1$, we have $x_i - 1 \geq 0$. Therefore, we have $f(x_1, x_2, \ldots, x_k) \in B$ and, thus, f is indeed a function from A to B.

- It is clear that f is one-to-one.
- For any $(y_1, y_2, \ldots, y_k) \in B$, we have

  $$y_1 + y_2 + \cdots + y_k = n - k,$$

  which is equivalent to

  $$(y_1 + 1) + (y_2 + 1) + \cdots + (y_k + 1) = n.$$

  Since each $y_i \geq 0$, we have $y_i + 1 \geq 1$. Therefore, $(y_1 + 1, y_2 + 1, \ldots, y_k + 1) \in A$ and

  $$f(y_1 + 1, y_2 + 1, \ldots, y_k + 1) = (y_1, y_2, \ldots, y_k).$$

  Thus, f is onto.

We have shown that f is a bijection.

Question 9: Let $n \geq 66$ be an integer and consider the set $S = \{1, 2, \ldots, n\}$.

- Let k be an integer with $66 \leq k \leq n$. How many 66-element subsets of S are there whose largest element is equal to k?

  Solution: Choosing a 66-element subset of S whose largest element is equal to k means that

  - k is included in the subset,
  - the 65 other elements in the subset are chosen from the set $\{1, 2, \ldots, k-1\}$.

  Therefore, the answer is $\binom{k-1}{65}$.

- Use the result in the first part to prove that

  $$\sum_{k=66}^{n} \binom{k-1}{65} = \binom{n}{66}.$$

  Solution: We are going to count the 66-element subsets of S in two different ways:

  - First way: use the definition of binomial coefficients; this gives as answer $\binom{n}{66}$.
  - Second way: We divide the 66-element subsets of S into groups, depending on the largest element in the subset. This gives, for each k with $66 \leq k \leq n$, the group G_k consisting of all 66-element subsets of S whose largest element is equal to k. We have seen in the first part that $|G_k| = \binom{k-1}{65}$. The groups are pairwise disjoint and their union contains all 66-element subsets of S. Therefore, the number of 66-element subsets of S is equal to

    $$\sum_{k=66}^{n} |G_k| = \sum_{k=66}^{n} \binom{k-1}{65}.$$