University of Ottawa - Department of Mathematics and Statistics
MAT 1322 D - Calculus II
Instructor: Petko Kitanov
October 14, 2015
Midterm Examination I
Version 2

Solutions

White

Name:.................................... Student Number:...........

Instructions :

- Please write your name and student number on the indicated area above.
- This is a closed book exam. It contains **6 questions**; there are 50 points in total.
- You can use non-programable and non-graphical calculators but no other aids are permitted.
- Clearly indicate the solution of each problem.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- If you need extra space, use the last two pages or the back of the pages.
- Time allowed: 80 minutes.

GOOD LUCK!

Student Number :____________________ Final Grade : ___________ out of 50

Question	1	2	3	4	5	6
Grade						

Question 1.

a) [4 pts] Evaluate the improper integral $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^{\frac{4}{3}}}\,dx$

b) [4 pts] Use the Comparison Theorem to determine whether the integral $\displaystyle\int_1^\infty \frac{3x\sin^2 x}{\sqrt{4x^6+3x}}\,dx$ is convergent or divergent.

(a) $$\int_e^\infty \frac{dx}{x(\ln x)^{4/3}} = \lim_{t\to\infty}\int_e^t \frac{dx}{x(\ln x)^{4/3}}$$

$$\int \frac{dx}{x(\ln x)^{4/3}} = \int (\ln x)^{-4/3}\,d(\ln x) = -3(\ln x)^{-1/3}$$

$$\lim_{t\to\infty}\int_e^t \frac{dx}{x(\ln x)^{4/3}} = \lim_{t\to\infty}\left[-\frac{3}{\sqrt[3]{\ln x}}\right]_e^t$$

$$= \lim_{t\to\infty}\left(-\frac{3}{\sqrt[3]{t}}\right) + 3 = 0+3 = 3$$

(b) $$\frac{3x\sin^2 x}{\sqrt{4x^6+3x}} \underset{\sin^2 x \le 1}{\le} \frac{3x}{\sqrt{4x^6+3x}} \underset{x\ge 1}{\le} \frac{3x}{\sqrt{4x^6}} = \frac{3x}{2x^3} = \frac{3}{2x^2}$$

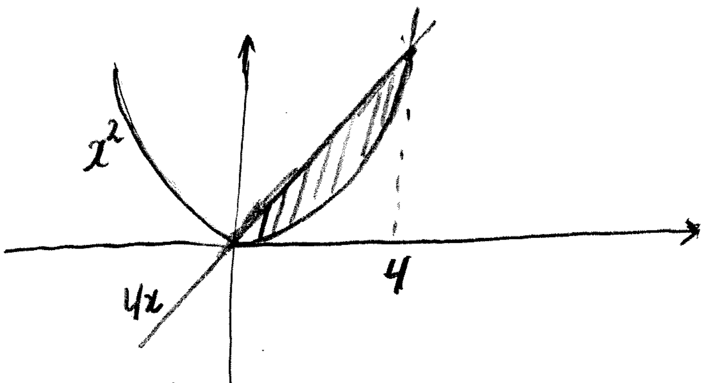
So, we have

$$\int_1^\infty \frac{3x\sin^2 x}{\sqrt{4x^6+3x}}\,dx \le \int_1^\infty \frac{3}{2x^2}\,dx$$

$$\underset{\text{convergent}}{\downarrow} \Longleftarrow \underset{\text{convergent}}{\downarrow}$$

Question 2. [10 pts]
a) Sketch the region enclosed by the curves $y = x^2$, $y = 4x$ and find its area.
b) Find the exact length of the curve $y = x^{\frac{3}{2}}$, $0 \leq x \leq 3$

(a)



$$x^2 = 4x$$
$$x(x-4) = 0$$
$$x = 0, \quad x = 4$$

$$A = \int_0^4 (4x - x^2)\,dx = \left[2x^2 - \frac{x^3}{3}\right]_0^4$$
$$= 32 - \frac{64}{3} = \frac{32}{3}$$

(b) $$L = \int_0^3 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\,dx \qquad \left(x^{3/2}\right)' = \frac{3}{2}x^{1/2}$$

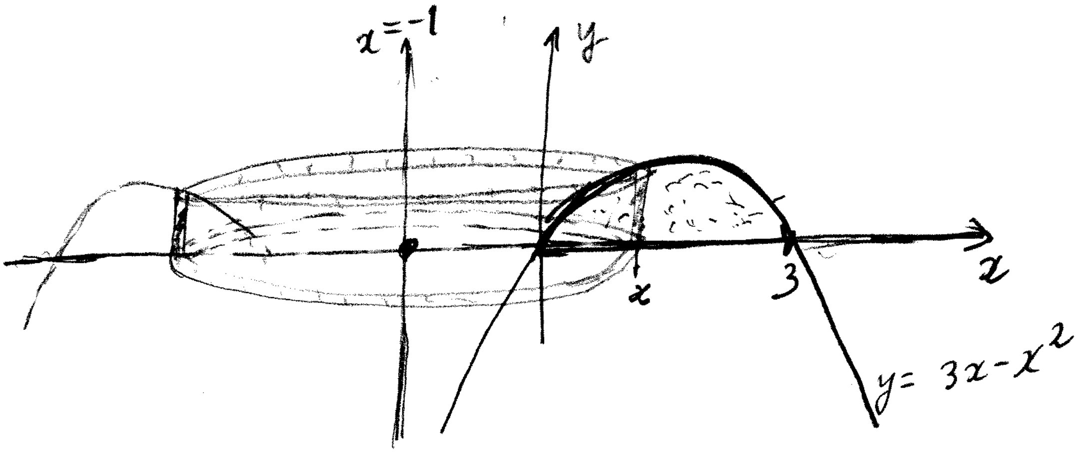
$$= \int_0^3 \sqrt{1 + \frac{9}{4}x}\,dx = \frac{4}{9}\int_0^3 \left(1 + \frac{9}{4}x\right)^{1/2} d\left(1 + \frac{9}{4}x\right)$$

$$= \frac{4}{9}\cdot\frac{2}{3}\left[1 + \frac{9}{4}x\right]^{3/2}\Bigg|_0^3 = \frac{8}{27}\left[\left(1 + \frac{27}{4}\right)^{3/2} - 1\right]$$

$$= \frac{8}{27}\left[\left(\frac{31}{4}\right)^{3/2} - 1\right] = \frac{8}{27}\left[\left(\frac{\sqrt{31}}{2}\right)^3 - 1\right]$$

Question 3. [10 pts] Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves $y = 3x - x^2$, $y = 0$, about the line $x = -1$.

$$V = \int_a^b 2\pi x\, f(x)\, dx$$



$$V = \int_0^3 2\pi (x+1)(3x-x^2)\, dx$$

$$= 2\pi \int_0^3 \left(-x^3+2x^2+3x\right) dx$$

$$= 2\pi \left[-\frac{x^4}{4} + \frac{2}{3}x^3 + \frac{3}{2}x^2\right]_0^3$$

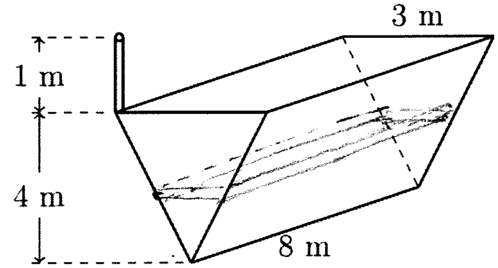
$$= 2\pi \left[-\frac{81}{4} + 18 + \frac{27}{2}\right]$$

$$= 2\pi \left[-\frac{81}{4} + \frac{72}{4} + \frac{54}{4}\right]$$

$$= 2\pi \frac{45}{4} = \frac{45}{2}\pi$$

Question 4. [10 pts] A reservoir in the form of a straight prism with triangular base is shown in the figure to the right.

Its vertical faces are isosceles triangles of height 4 m and base 3 m, its length is 8 m, it is near the surface of the Earth, and it is full of water, which will be pumped to a height of 1 m above the reservoir.



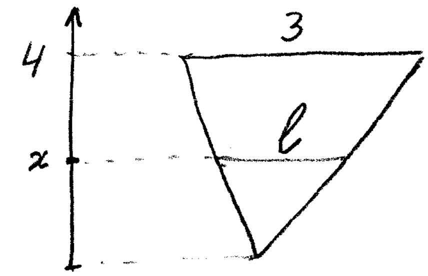
Denote by x the height in meters measured **from the bottom of the reservoir**.

a) What is, at first approximation, the volume of the layer of water between the heights x and $x + \Delta x$?

b) What is, at first approximation, the work required to pump that layer of water to a height of 1 m above the reservoir? Recall that the density of water is 1000 kg/m^3, and gravitational acceleration at the surface of the earth is $g \cong 9.8$ m/s^2.

c) What is, in Joules, the work required to pump all the water from the reservoir to a height of 1 m above the reservoir?

(a) A horizontal cross section at height x is a rectangle with length 8 and width l. Using similar triangles, we find l



$$\frac{l}{x} = \frac{3}{4} \Rightarrow l = \frac{3x}{4}$$

The volume of the layer $\Delta V \approx$ (area of the layer) $\times$ (Δx)

$$\Delta V = 8 \cdot \frac{3x}{4} \Delta x = 6x\Delta x$$

(b) Work $\approx$ Force $\times$ Distance $= 1000g(\Delta V) \times$ (distance)

$$\Delta W \approx 1000 \cdot (9.8)(6x\Delta x)(5-x) = 58800x(5-x)\Delta x$$

(c)
$$W = \int_0^4 58800x(5-x)dx = 58800\int_0^4 (5x-x^2)dx$$

$$= 58800\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^4 = 58800\left(\frac{5 \cdot 16}{2} - \frac{64}{3}\right)$$

$$= 1{,}097{,}600$$

Question 5. [4 pts] Use Euler's method with step $h = 0.1$ to estimate $y(0.2)$, where $y(x)$ is the solution of the initial value problem $y' = y - 2x$, $\quad y(0) = 1$.

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$x_0 = 0 \ , \ y_0 = 1$$

$$x_1 = 0.1$$

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= 1 + (0.1)(y_0 - 2x_0)$$
$$= 1 + (0.1)(1 - 2 \cdot 0)$$
$$= 1.1$$

$$x_1 = 0.1 \ , \ x_2 = 0.2$$

$$y_2 = y_1 + h f(x_1, y_1)$$
$$= 1.1 + (0.1)(1.1 - 2(0.1))$$
$$= 1.1 + (0.1)(0.9)$$
$$= 1.1 + 0.09$$
$$= 1.19$$

Question 6.

a) [3 pts] Solve the differential equation $y' + 2xy^2 = 0$. Write down the solution in explicit form.

b) [5 pts] Solve the initial value problem $\frac{dy}{dx} = 3(y^2+1)$, $y(\pi/3) = 1$.

(a) $\frac{dy}{dx} = -2xy^2$

$\int \frac{dy}{y^2} = \int -2x\,dx + C$

$-\frac{1}{y} = -x^2 + C$ $\qquad \frac{1}{y} = x^2 - C$

$y = \frac{1}{x^2 - C}$

(b) $\frac{dy}{dx} = 3(y^2+1)$

$\int \frac{dy}{y^2+1} = \int 3\,dx + C$

$\arctan y = 3x + C$

$y = \tan(3x + C)$ General solution

$y(\pi/3) = \tan(\pi + C) = 1$

$\Rightarrow \pi + C = \frac{\pi}{4}$

$C = -\frac{3}{4}\pi$

Solution of the IVP is

$y = \tan\left(3x - \frac{3\pi}{4}\right)$