Q1 [9 marks]

Find the following limits.

(a) $\lim_{x\to 2} \frac{x^2-3x+2}{x^2-4}$ $= \lim_{x\to 2} \frac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}(x+2)}$

$= \frac{1}{4}.$

(b) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$ $= \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$

$= \frac{1}{2}.$

(c) If $\lim_{x\to 1} f(x) = 8$ and $\lim_{x\to 1} g(x) = 3$, then find $\lim_{x\to 1} \sqrt[3]{f(x)g(x)+3}$.

$\lim_{x\to 1} \sqrt[3]{f(x)g(x)+3} = \sqrt[3]{\lim_{x\to 1} \cancel{f(x)}g(x)+3}$

$= \sqrt[3]{\lim_{x\to 1} f(x) \cdot \lim_{x\to 1} g(x) + 3}$

$= \sqrt[3]{8 \cdot 3 + 3}$

$= 3$

Q2 [9 marks]

Compute the derivatives of the following functions. DO NOT SIMPLIFY.

(a) $f(x) = \dfrac{x^2 + 12x + e^3}{x + e^x}$

$$f'(x) = \frac{(2x+12)(x+e^x) - (x^2+12x+e^3)(1+e^x)}{(x+e^x)^2}$$

(b) $g(t) = e^{3t}(t^2 + x^2)$ x is a constant

$$g'(t) = 3e^{3t}(t^2+x^2) + e^{3t} \cdot 2t.$$

(c) $f(x) = (x^2 + x + 1)(x^3 + 1)^3$

$$f'(x) = (2x+1)(x^3+1)^3 + (x^2+x+1) \cdot 3(x^3+1)^2 \cdot 3x^2$$

Q3 [7 marks]

(*a*) (2 marks) Carefully state the definition of the derivative of a function $f(x)$ at a point $x = a$.

A function $f(x)$ is differentiable at $x = a$ if and only if $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. [Alternately, $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$]

(*b*) (5 marks) Use the definition of the derivative from part (a) to compute $f'(1)$ for $f(x) = \dfrac{13}{x+7}$. NO CREDIT will be given for any other method.

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{\frac{13}{x+7} - \frac{13}{8}}{x - 1}$$

$$= \lim_{x \to 1} \frac{\frac{13 \cdot 8 - 13(x+7)}{8(x+7)}}{x - 1}$$

$$= \lim_{x \to 1} \frac{-13\cancel{(x-1)}}{\cancel{(x-1)} \cdot 8 \cdot (x+7)} = -\frac{13}{64}$$

Q4 [5 marks]

Find the values of the parameters a and b such that the function

$$f(x) = \begin{cases} (2x+a)^3, & \text{if } x \leq 0, \\ 5bx + 8, & \text{if } 0 < x \leq 1, \\ x^2 + 12, & \text{if } x > 1, \end{cases}$$

is continuous at all the points in its domain. Is f differentiable at all points in its domain with these values of a and b?

We focus on $x=0$ and $x=1$ since f is continuous at all other points because polynomials are continuous.

At $x=0$: Forcing $\lim_{x \to 0^-} (2x+a)^3 = \lim_{x \to 0^+} 5bx+8$

$$\Rightarrow \quad a^3 = 8 \Rightarrow \boxed{a=2}.$$

At $x=1$: Forcing $\lim_{x \to 1^-} (5bx+8) = \lim_{x \to 1^+} x^2+12$

$$\Rightarrow \quad 5b+8 = 13 \Rightarrow \boxed{b=1}.$$

Now, at $x=0$, we ask if the left and right derivatives are equal: $3(2x+2)^2 \cdot 2\Big|_{x=0} \stackrel{?}{=} 5\Big|_{x=0}$

$$\Rightarrow \quad 24 \stackrel{?}{=} 5 \quad \underline{\text{No}}!$$

Hence f is not differentiable at $x=0$, so it is not differentiable at all points in the domain.

Q5 [8 marks]

Find the equation of the tangent line to the curve $y = f(x) = \frac{1}{\sqrt[3]{x^2}}$ that is parallel to the line $y - 2x = \pi$.

The slope of $y - 2x = \pi$ is 2, so we want $(a, f(a))$ where $f'(a) = 2$:

$$f'(x) = -\frac{2}{3} x^{-5/3}$$

$\Rightarrow$ we solve $-\frac{2}{3} a^{-5/3} = 2$

$$\Rightarrow a^{-5/3} = -3$$

$$\Rightarrow a = -3^{-3/5} = \frac{-1}{\sqrt[5]{27}}.$$

This gives $f(a) = f(-3^{-3/5}) = 3^{2/5} = \sqrt[5]{9}.$

So, the tangent line at this point is

$$\boxed{y - \sqrt[5]{9} = 2\left(x + \frac{1}{\sqrt[5]{27}}\right)}.$$

Q6 [12 marks]

When EZ Electronics Company sells surge protectors at \$50 a piece, they produce and sell 3000 of them per month. For every \$1 increase in price, the number of surge protectors they sell decreases by 15. Assume that the fixed production costs are \$50,000 and the variable costs are \$30 per surge protector.

(a) Find the linear demand function $q = D(p)$, where p is a price of a unit and q is the number of surge protectors made and sold. [Hint: The point $(p, q) = (50, 3000)$ must lie on this line.]

$$\frac{q - q_0}{p - p_0} = m = -15$$

$$\Rightarrow \quad q - 3000 = -15(p - 50)$$

$$\Rightarrow \quad q = -15p + 3750.$$

(b) Find the cost function $C(q)$ as a function of q, and then express it as a function of p.

$$C(q) = 50\,000 + 30q$$
$$= 50\,000 + 30(-15p + 3750)$$
$$= -450p + 162\,500.$$

(c) Find the revenue function $R(q)$ as a function of q, and then express it as a function of p.

$$R = p \cdot q(p) = p(-15p + 3750)$$

$$R(p) = -15p^2 + 3750p.$$

(*d*) Find the marginal profit, $MP(p)$, with respect to p.

Profit = Revenue − Costs

$\Rightarrow \quad P(p) = -15p^2 + 3750p - (-450p + 162500)$

$\Rightarrow \quad MP(p) = P'(p) = -30p + 4200$

(*e*) Find the *break-even points*. Give both the price p and quantity q at each of these points.

Break-even: $R(p) = C(p)$

$\Rightarrow -15p^2 + 3750p = -450p + 162500$

$\Rightarrow p = \$46.37$ or $p = \$233.63$.

Points: $(46.37, 3054)$ AND $(233.63, 245)$ are Break-even pts.

(*f*) If EZ Electronics Company is operating at the higher break-even point, should it increase or decrease the price of its surge protectors to increase its profits? Explain your answer.

"Higher" here means higher price p.

At $p = 233.63$, $MP(233.63) < 0$,

so increasing the price p lowers ~~p~~ profits

and decreasing price raises profits.

The company should LOWER the price

to increase profits.