Example of a Sturm-Liouville Problem

Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$y'' - y' + \lambda y = 0$$

$$y(0) = 0 = y(L).$$

Solution: To solve the ODE we look at the auxiliary equation

$$r^2 - r + \lambda = 0.$$

It has roots

$$r = \frac{1 \pm \sqrt{1-4\lambda}}{2}.$$

The type of solutions we get will depend on whether $1 - 4\lambda$ is positive, negative or 0. We consider these three situations separately.
CASE 1: $1 - 4\lambda = 0$. In this case the two roots are $r = \frac{1}{2}, \frac{1}{2}$. Since we have repeated roots the solution to the ODE is

$$y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}.$$

Plugging in the first boundary condition we see

$$y(0) = c_1 = 0.$$

Thus $c_1 = 0$. The other boundary condition then gives

$$y(L) = c_2 L e^{\frac{1}{2}L} = 0.$$

Since $Le^{\frac{1}{2}L}$ is not zero we can divide by it to see that $c_2 = 0$. Thus the only solution to the Sturm-Liouville problem we get is the trivial one $y = 0$. We ignore these.
CASE 2: $1 - 4\lambda > 0$. In this case we have two real roots $r = \frac{1+\sqrt{1-4\lambda}}{2}$ and $\frac{1-\sqrt{1-4\lambda}}{2}$. Thus our solution to the ODE look like

$$y = c_1 e^{\frac{1+\sqrt{1-4\lambda}}{2}x} + c_2 e^{\frac{1-\sqrt{1-4\lambda}}{2}x}.$$

Plugging in the first boundary condition we see

$$y(0) = c_1 + c_2 = 0.$$

So $c_1 = -c_2$. Plugging in the second boundary condition now yields

$$y(L) = c_1\left(e^{\frac{1+\sqrt{1-4\lambda}}{2}L} - e^{\frac{1-\sqrt{1-4\lambda}}{2}L}\right) = 0.$$

Since the term in the parenthesis is non-zero (why is it non-zero!) we see that $c_1 = 0$ and thus $c_2 = 0$. So our only solution in this case is the trivial one $y = 0$.

CASE 3: $1 - 4\lambda < 0$. In this case we get complex roots $r = \frac{1}{2} \pm i\frac{\sqrt{4\lambda-1}}{2}$. So the solution to the ODE is

$$y = c_1 e^{\frac{1}{2}x} \sin(\frac{\sqrt{4\lambda - 1}}{2}x) + c_2 e^{\frac{1}{2}x} \cos(\frac{\sqrt{4\lambda - 1}}{2}x).$$

Plugging in the fist boundary condition gives

$$y(0) = c_2 = 0.$$

So $c_2 = 0$. Now plugging in the second boundary condition gives

$$y(L) = c_1 e^{\frac{1}{2}L} \sin(\frac{\sqrt{4\lambda - 1}}{2}L) = 0.$$

In order not to have a trivial solution we assume $c_1 \neq 0$ then this equation implies

$$\sin(\frac{\sqrt{4\lambda - 1}}{2}L) = 0.$$

Recall the zeros of sin occur at integral multiples of π. Thus to satisfy the last boundary condition we must have

$$\frac{\sqrt{4\lambda - 1}}{2}L = n\pi$$

for some integer n (since we are in case 3 we need $n > 0$). The λ's that satisfy this equation are

$$\lambda_n = \frac{(2n\pi)^2 + L^2}{4L^2}.$$

And the eigenfunction corresponding to λ_n is

$$e^{\frac{1}{2}x} \sin\frac{n\pi}{L}x.$$

Answer:

$$\text{eigenvalues } \lambda_n = \frac{(2n\pi)^2 + L^2}{4L^2} \quad \text{for } n = 1, 2, \ldots$$

$$\text{eigenfunctions} y_n = e^{\frac{1}{2}x} \sin\frac{n\pi}{L}x \quad \text{for } n = 1, 2, \ldots$$