Math 327
Exam 2 - Practice Problem Solutions

1. For each of the following matrices, determine whether it is in row echelon form, reduced row echelon form, or neither.

(a) $\begin{bmatrix} 1 & -4 & 2 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

Since each row has a leading 1 that is down and to the right of the leading 1 in the previous row, this matrix is in row echelon form.
Since some of the columns with a leading 1 have other non-zero entries, it is not in reduced row echelon form.

(b) $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{bmatrix}$

Since each non-zero row has a leading 1 that is down and to the right of the leading 1 in the previous row, each column with a leading 1 has no other non-zero entries, and the zero rows is at the bottom of the matrix, this matrix is in reduced row echelon form.

(c) $\begin{bmatrix} 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

Since the last row is not a zero row but does not have a leading 1, this matrix is in neither row echelon form nor reduced row echelon form.

2. Put each of the following matrices into *row echelon form*.

(a) $\begin{bmatrix} 3 & -2 & 4 & 7 \\ 2 & 1 & 0 & -3 \\ 2 & 8 & -8 & 2 \end{bmatrix}$

$$\begin{bmatrix} 3 & -2 & 4 & 7 \\ 2 & 1 & 0 & -3 \\ 2 & 8 & -8 & 2 \end{bmatrix} \underset{r_3 - r_2 \to r_3}{\overset{r_1 - r_2 \to r_1}{}} \begin{bmatrix} 1 & -3 & 4 & 10 \\ 2 & 1 & 0 & -3 \\ 0 & 7 & -8 & 5 \end{bmatrix} r_2 - 2r_1 \to r_1 \begin{bmatrix} 1 & -3 & 4 & 10 \\ 0 & 7 & -8 & -23 \\ 0 & 7 & -8 & 5 \end{bmatrix}$$

$$\underset{\frac{1}{7}r_2 \to r_2}{\overset{r_3 - r_2 \to r_3}{}} \begin{bmatrix} 1 & -3 & 4 & 10 \\ 0 & 1 & -\frac{8}{7} & -\frac{23}{7} \\ 0 & 0 & 0 & 28 \end{bmatrix} \frac{1}{28}r_3 \to r_3 \begin{bmatrix} 1 & -3 & 4 & 10 \\ 0 & 1 & -\frac{8}{7} & -\frac{23}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) $\begin{bmatrix} 3 & -1 & 2 & 4 & 1 \\ 2 & 1 & 3 & -1 & 2 \\ 1 & 2 & 3 & -2 & 3 \end{bmatrix}$

$$\begin{bmatrix} 3 & -1 & 2 & 4 & 1 \\ 2 & 1 & 3 & -1 & 2 \\ 1 & 2 & 3 & -2 & 3 \end{bmatrix} r_1 \leftrightarrow r_3 \begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 2 & 1 & 3 & -1 & 2 \\ 3 & -1 & 2 & 4 & 1 \end{bmatrix} \underset{r_3 - 3r_1 \to r_3}{\overset{r_2 - 2r_1 \to r_2}{}} \begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 0 & -3 & -3 & 3 & -4 \\ 0 & -7 & -7 & 10 & 8 \end{bmatrix}$$

$$-\frac{1}{3}r_2 \to r_2 \begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 0 & 1 & 1 & -1 & \frac{4}{3} \\ 0 & -7 & -7 & 10 & 8 \end{bmatrix} r_3 + 7r_2 \to r_3 \begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 0 & 1 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 3 & \frac{4}{3} \end{bmatrix} \frac{1}{3}r_3 \to r_3 \begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 0 & 1 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 & \frac{4}{9} \end{bmatrix}$$

(c) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \frac{1}{\cos\theta}r_1 \to r_1 \begin{bmatrix} 1 & \frac{\sin\theta}{\cos\theta} \\ -\sin\theta & \cos\theta \end{bmatrix} r_2 + \sin\theta r_1 \to r_2 \begin{bmatrix} 1 & \frac{\sin\theta}{\cos\theta} \\ 0 & \frac{\sin^2\theta}{\cos\theta} + \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & \frac{\sin\theta}{\cos\theta} \\ 0 & \frac{\sin^2\theta}{\cos\theta} + \frac{\cos^2\theta}{\cos\theta} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{\sin\theta}{\cos\theta} \\ 0 & \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} \end{bmatrix} = \begin{bmatrix} 1 & \tan\theta \\ 0 & \frac{1}{\cos\theta} \end{bmatrix} \cos\theta r_2 \to r_2 \begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$$

3. Put each of the matrices from the previous problem into reduced row echelon form.

(a) Continuing from above:

$$\begin{bmatrix} 1 & -3 & 4 & 10 \\ 0 & 1 & -\frac{8}{7} & -\frac{23}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix} r_1 + 3r_2 \to r_1 \begin{bmatrix} 1 & 0 & \frac{4}{7} & \frac{1}{7} \\ 0 & 1 & -\frac{8}{7} & -\frac{23}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{r_2 + \frac{23}{7}r_3 \to r_2}{r_1 - \frac{1}{7}r_3 \to r_1} \begin{bmatrix} 1 & 0 & \frac{4}{7} & 0 \\ 0 & 1 & -\frac{8}{7} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Continuing from above:

$$\begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 0 & 1 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 & \frac{4}{9} \end{bmatrix} \overset{r_1 - 2r_2 \to r_1}{r_2 + r_3 \to r_2} \begin{bmatrix} 1 & 0 & 1 & 0 & \frac{1}{3} \\ 0 & 1 & 1 & 0 & \frac{16}{9} \\ 0 & 0 & 0 & 1 & \frac{4}{9} \end{bmatrix}$$

(c) Continuing from above:

$$\begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix} r_1 - \tan\theta r_2 \to r_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. Use Gaussian Elimination to find all solutions to the following systems of linear equations.

(a) $\begin{cases} x + 2y + 3z = 9 \\ 2x - 2z = -2 \\ 3x + 2y + z = 7 \end{cases}$

We begin by finding the augmented matrix associated with this system of equations and then carry out row operations:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & 0 & 2 & -2 \\ 3 & 2 & 1 & 7 \end{array}\right] \overset{r_2 - 2r_1 \to r_2}{r_3 - 3r_1 \to r_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \\ 0 & -4 & -8 & -20 \end{array}\right] \overset{r_3 - r_2 \to r_3}{-\frac{1}{4}r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then $y + 2z = 5$, so $y = 5 - 2z$, and $x + 2y + 3z = 9$, or, substituting, $x + 2(5 - 2z) + 3z = 9$. Therefore, $x = 10 - 4z + 3z = 9$, so $x = -1 + z$

Hence the solutions to this system are all of the form $(-1 + t, 5 - 2t, t)$ for some $t \in \mathbb{R}$.

(b) $\begin{cases} x + 2y + 3z = 9 \\ 3x + 2y + z = 7 \end{cases}$

We begin by finding the augmented matrix associated with this system of equations and then carry out row operations:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 3 & 2 & 1 & 7 \end{array}\right] r_3 - 3r_1 \to r_3 \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \end{array}\right] -\frac{1}{4}r_2 \to r_2 \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 5 \end{array}\right]$$

Then, exactly as above, $y + 2z = 5$, so $y = 5 - 2z$, and $x + 2y + 3z = 9$, or, substituting, $x + 2(5 - 2z) + 3z = 9$. Therefore, $x = 10 - 4z + 3z = 9$, so $x = -1 + z$

Hence the solutions to this system are all of the form $(-1 + t, 5 - 2t, t)$ for some $t \in \mathbb{R}$.

5. Use Gauss-Jordan reduction to find all solutions to the following systems of linear equations.

(a) $\begin{cases} 3x - 2y + z = -6 \\ 4x - 3y + 3z = 7 \\ 2x + y - z = -9 \end{cases}$

We begin by finding the augmented matrix associated with this system of equations and then carry out row operations:

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & -6 \\ 4 & -3 & 3 & 7 \\ 2 & 1 & -1 & -9 \end{array}\right] \overset{r_2 - r_1 \to r_1}{2r_3 - r_2 \to r_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 13 \\ 4 & -3 & 3 & 7 \\ 0 & 5 & -5 & -25 \end{array}\right] \overset{r_2 - 4r_1 \to r_2}{\frac{1}{5}r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 13 \\ 0 & -1 & 5 & 45 \\ 0 & 1 & -1 & -5 \end{array}\right]$$

$$\xrightarrow[r_3+r_2\to r_3]{r_1+r_3\to r_1}\left[\begin{array}{ccc|c}1&0&1&8\\0&-1&5&45\\0&0&4&40\end{array}\right]\xrightarrow[\frac{1}{4}r_3\to r_3]{-r_2\to r_2}\left[\begin{array}{ccc|c}1&0&1&8\\0&1&-5&-45\\0&0&1&10\end{array}\right]\xrightarrow[r_2+5r_3\to r_2]{r_1-r_3\to r_1}\left[\begin{array}{ccc|c}1&0&0&-2\\0&1&0&5\\0&0&1&10\end{array}\right]$$

Therefore, the unique solution is $x = -2$, $y = 5$, and $z = 10$

(b) $\begin{cases} 3x - 2y + z = 4 \\ x + 3y - z = -3 \\ 4x - 10y + 4z = 10 \end{cases}$

We begin by finding the augmented matrix associated with this system of equations and then carry out row operations:

$$\left[\begin{array}{ccc|c}3&-2&1&4\\1&3&-1&-3\\4&-10&4&10\end{array}\right]\xrightarrow{r_1\leftrightarrow r_2}\left[\begin{array}{ccc|c}1&3&-1&-3\\3&-2&1&4\\4&-10&4&10\end{array}\right]\xrightarrow[r_3-4r_1\to r_3]{r_2-3r_1\to r_2}\left[\begin{array}{ccc|c}1&3&-1&-3\\0&-11&4&13\\0&-22&8&22\end{array}\right]$$

$$\xrightarrow{r_3-2r_2\to r_3}\left[\begin{array}{ccc|c}1&3&-1&-3\\0&-11&4&13\\0&0&0&-4\end{array}\right]$$

Therefore, this system has no solutions (the equation $0 = -4$ is inconsistent).

6. Use quadratic interpolation to find a quadratic function passing through the points $(0, 2)$, $(1, 0)$, and $(2, 4)$.

If we start with the general quadratic $p(x) = ax^2 + bx + c$ and the three conditions given above, we have the following system of equations:

$$\begin{cases} c = 2 \\ a + b + c = 0 \\ 4a + 2b + c = 4 \end{cases}$$

In matrix form, this is: $\left[\begin{array}{ccc|c}0&0&1&2\\1&1&1&0\\4&2&1&4\end{array}\right]\xrightarrow[r_2\leftrightarrow r_3]{r_1\leftrightarrow r_2}\left[\begin{array}{ccc|c}1&1&1&0\\4&2&1&4\\0&0&1&2\end{array}\right]\xrightarrow[r_1-r_3\to r_1]{4r_1-r_2\to r_2}\left[\begin{array}{ccc|c}1&1&0&-2\\0&2&3&-4\\0&0&1&2\end{array}\right]$

$$\xrightarrow{r_2-3r_3\to r_2}\left[\begin{array}{ccc|c}1&1&0&-2\\0&2&0&-10\\0&0&1&2\end{array}\right]\xrightarrow{\frac{1}{2}r_2\to r_2}\left[\begin{array}{ccc|c}1&1&0&-2\\0&1&0&-5\\0&0&1&2\end{array}\right]\xrightarrow{r_1-r_2\to r_1}\left[\begin{array}{ccc|c}1&0&0&3\\0&1&0&-5\\0&0&1&2\end{array}\right]$$

Therefore, we have $a = 3$, $b = -5$, and $c = 2$, so $p(x) = 3x^2 - 5x + 2$. Once can quickly verify that this quadratic passes through all three desired points.

7. Construct a quadratic polynomial $p(x) = ax^2 + bx + c$ satisfying the conditions: $p(1) = f(1)$, $p'(1) = f'(1)$, and $p''(1) = f''(1)$ if $f(x) = x\ln x + x^2$.

First, notice that since $f(x) = x\ln x + x^2$, then $f'(x) = \ln x + x\left(\frac{1}{x}\right) + 2x = \ln x + 2x + 1$ and $f''(x) = \left(\frac{1}{x}\right) + 2$. Then $f(1) = 1$, $f'(1) = 3$, and $f''(1) = 3$.

Next, notice that since $p(x) = ax^2 + bx + c$, then $p'(x) = 2ax + b$ and $p''(x) = 2a$. Then $p(1) = a + b + c$, $p'(1) = 2a + b$, and $p''(x) = 2a$

This gives the system of equations: $\begin{cases} a + b + c = 1 \\ 2a + b = 3 \\ 2a = 3 \end{cases}$

The matrix form of this equation is: $\left[\begin{array}{ccc|c}1&1&1&1\\2&1&0&3\\2&0&0&3\end{array}\right]\xrightarrow[\frac{1}{2}r_2]{r_2-r_3\to r_2}\left[\begin{array}{ccc|c}1&1&1&1\\0&1&0&0\\1&0&0&\frac{3}{2}\end{array}\right]\xrightarrow[r_1-r_3\to r_1]{r_1-r_2\to r_1}\left[\begin{array}{ccc|c}0&0&1&-\frac{1}{2}\\0&1&0&0\\1&0&0&\frac{3}{2}\end{array}\right]$

Thus, we have $a = \frac{3}{2}$, $b = 0$, and $c = -\frac{1}{2}$. Hence $p(x) = \frac{3}{2}x^2 - \frac{1}{2}$. One can verify this this quadratic satisfies all of the given conditions.

8. Find the elementary matrices corresponding to carrying out each of the following elementary row operations on a 3×3 matrix:

(a) $r_2 \leftrightarrow r_3$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) $-\frac{1}{4}r_2 \to r_2$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $3r_1 + r_2 \to r_2$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Find the inverse of each of the elementary matrices you found in the previous problem.

(a) $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(b) $E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. Write the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ as a product of elementary matrices. [Hint: what elementary operations are needed to put A into reduced row echelon form]

We begin by putting A into r.r.e.f.:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} r_2 - 3r_1 \to r_2 \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} r_1 + r_2 \to r_1 \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} -\tfrac{1}{2}r_2 \to r_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Next, we observe that the elementary matrices used in this reduction process are: $E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$.

Then $E_3E_2E_1A = I_2$, so $A^{-1} = E_3E_2E_1$. Hence $A = \left(A^{-1}\right)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$.

Thus $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

11. For each of the following matrices, either find the inverse or prove that the matrix is singular.

(a) $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] r_1 - r_2 \to r_1 \left[\begin{array}{cc|cc} 1 & -5 & 1 & -1 \\ 2 & 4 & 0 & 1 \end{array}\right] r_2 - 2r_1 \to r_2 \left[\begin{array}{cc|cc} 1 & -5 & 1 & -1 \\ 0 & 14 & -2 & 3 \end{array}\right] \tfrac{1}{4}r_2 \to r_2 \left[\begin{array}{cc|cc} 1 & -5 & 1 & -1 \\ 0 & 1 & -\frac{1}{7} & \frac{3}{14} \end{array}\right]$$

$$r_1 + 5r_2 \to r_1 \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{7} & \frac{1}{14} \\ 0 & 1 & -\frac{1}{7} & \frac{3}{14} \end{array}\right]$$

Thus $A^{-1} = \begin{bmatrix} \frac{2}{7} & \frac{1}{14} \\ -\frac{1}{7} & \frac{3}{14} \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -5 \\ -1 & \frac{5}{3} \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 3 & -5 & 1 & 0 \\ -1 & \frac{5}{3} & 0 & 1 \end{array}\right] \underset{r_1 \leftrightarrow r_2}{\overset{-r_1 \to r_1}{}} \left[\begin{array}{cc|cc} 1 & -\frac{5}{3} & 0 & -1 \\ 3 & -5 & 1 & 0 \end{array}\right] r_2 - 3r_1 \to r_2 \left[\begin{array}{cc|cc} 1 & -\frac{5}{3} & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Since we encountered a row of zeros, A is a singular matrix, so A^{-1} does not exist.

(c) $\begin{bmatrix} -3 & 0 & 4 \\ 2 & -1 & 0 \\ 5 & 0 & -2 \end{bmatrix}$

The standard row reduction algorithm yields: $A^{-1} = \begin{bmatrix} \frac{1}{7} & 0 & \frac{2}{7} \\ \frac{2}{7} & -1 & \frac{4}{7} \\ \frac{5}{14} & 0 & \frac{3}{14} \end{bmatrix}$

(d) $\begin{bmatrix} -3 & 0 & 4 \\ 2 & -1 & 0 \\ 5 & -1 & 4 \end{bmatrix}$

The standard row reduction algorithm yields: $A^{-1} = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & -\frac{4}{3} & \frac{1}{3} \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \end{bmatrix}$

12. Find all values of a for which the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & a \\ 0 & 2 & -a \end{bmatrix}$ has an inverse. Find the form of A^{-1} (in terms of a) for the cases where it exists.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & a & 0 & 1 & 0 \\ 0 & 2 & -a & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \scriptstyle r_2-2r_1\to r_2 \\ \frac{1}{2}r_3 \to r_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & a-4 & -2 & 1 & 0 \\ 0 & 1 & -\frac{a}{2} & 0 & 0 & \frac{1}{2} \end{array}\right] \begin{array}{c} \scriptstyle r_2\leftrightarrow r_3 \\ \frac{1}{a-4}r_3 \to r_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{a}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{2}{a-4} & \frac{1}{a-4} & 0 \end{array}\right]$$

$$\begin{array}{c} \scriptstyle r_1-2r_3\to r_1 \\ r_2 + \frac{a}{2}r_3 \to r_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{a}{a-4} & -\frac{2}{a-4} & 0 \\ 0 & 1 & 0 & -\frac{a}{a-4} & \frac{a}{2(a-4)} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{2}{a-4} & \frac{1}{a-4} & 0 \end{array}\right]$$

Thus $A^{-1} = \begin{bmatrix} \frac{a}{a-4} & -\frac{2}{a-4} & 0 \\ -\frac{a}{a-4} & \frac{a}{2(a-4)} & \frac{1}{2} \\ -\frac{2}{a-4} & \frac{1}{a-4} & 0 \end{bmatrix}$. Notice that we must have $a \neq 4$ for the inverse to exist.

13. Given the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ -1 & 2 & -5 \end{bmatrix}$:

(a) Find matrix B equivalent to A having the form described by Theorem 2.12

We will solve both parts of this question at the same time by utilizing matrices of the form $[A|I_n]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ -1 & 2 & -5 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \scriptstyle r_2-r_1\to r_2 \\ r_3 + r_1 \to r_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 3 & -3 & 1 & 0 & 1 \end{array}\right] \begin{array}{c} \scriptstyle r_1-r_2\to r_1 \\ r_3 - 3r_2 \to r_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 4 & -3 & 1 \end{array}\right]$$

Restarting our right matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] c_3 - 3c_1 \to c_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] c_3 + c_2 \to c_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Then the reduced form of A is: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) Find matrices P and Q such that $B = PAQ$ (for the matrices A and B from above).

Using the right hand sides of the reductions performed above (the row and column reductions respectively) we see that:

$P = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

14. For each permutation given below, first find the number of inversions. Then, classify the permutation as either even or odd.

(a) 53214

Notice that this permutation has 7 inversions, so it is an odd permutation.

(b) 45132

Notice that this permutation has 7 inversions, so it is an odd permutation.

(c) 34152

Notice that this permutation has 5 inversions, so it is an odd permutation.

15. Find the determinant of each of the following matrices:

(a) $\begin{bmatrix} 2 & -4 \\ 5 & 1 \end{bmatrix}$

$$\begin{vmatrix} 2 & -4 \\ 5 & 1 \end{vmatrix} = (2)(1) - (-4)(5) = 2 + 20 = 22.$$

(b) $\begin{bmatrix} 3 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 8 & -5 \end{bmatrix}$

$$\begin{vmatrix} 3 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 8 & -5 \end{vmatrix} = (-15) + (0) + (-32) - (0) - (96) = -143.$$

(c) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

$$\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix} = a_{31}A_{31} + 0 + 0 + 0 = 4\begin{vmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -3 \end{vmatrix} = 4[0 + 0 + 0 - 0 - 0 - 6] = -24$$

(d) $\begin{bmatrix} t & 3 \\ 2 & t+1 \end{bmatrix}$

$$\begin{vmatrix} t & 3 \\ 2 & t+1 \end{vmatrix} = t(t+1) - 6 = t^2 + t - 6 = (t+3)(t-2)$$

16. For which values of t is the matrix in part (d) above singular? Justify your answer.

Recall that a matrix A is singular if and only if $det(A) = 0$. For this matrix, $det(A) = t(t+1) - 6 = t^2 + t - 6 = (t+3)(t-2)$.

Hence this matrix is singular when $t = -3$ and $t = 2$.

17. Evaluate each of the following:

(a) $\begin{vmatrix} 4 & -2 & 3 & -5 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & -1 & 18 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

$$\begin{vmatrix} 4 & -2 & 3 & -5 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & -1 & 18 \\ 0 & 0 & 0 & 2 \end{vmatrix} = (4)(1)(-1)(2) = -8$$

(b) $\begin{bmatrix} t-1 & -1 & -2 \\ 0 & t-2 & 2 \\ 0 & 0 & t-3 \end{bmatrix}$

$$\begin{bmatrix} t-1 & -1 & -2 \\ 0 & t-2 & 2 \\ 0 & 0 & t-3 \end{bmatrix} = (t-1)(t-2)(t-3) = t^3 - 6t^2 + 11t - 6$$

(c) $\begin{vmatrix} 4 & -3 & 6 & 1 \\ 4 & 0 & 4 & 1 \\ 4 & 2 & 0 & 1 \\ 4 & 1 & 2 & 1 \end{vmatrix}$

$$\begin{vmatrix} 4 & -3 & 6 & 1 \\ 4 & 0 & 4 & 1 \\ 4 & 2 & 0 & 1 \\ 4 & 1 & 2 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & -3 & 6 & 1 \\ 1 & 0 & 4 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{vmatrix} = 0 \text{ (Notice that the second matrix has a repeated column).}$$

18. Compute the determinant of the following matrix by putting it into triangular form: $A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 3 & 2 & -4 & -2 \\ 2 & 0 & 1 & 4 \\ 3 & 8 & -8 & -10 \end{bmatrix}$

$$\begin{vmatrix} 2 & 3 & -1 & 0 \\ 3 & 2 & -4 & -2 \\ 2 & 0 & 1 & 4 \\ 3 & 8 & -8 & -10 \end{vmatrix} \overset{r_3 - r_1 \to r_3}{r_4 - r_2 \to r_2 =} \begin{vmatrix} 2 & 3 & -1 & 0 \\ 3 & 2 & -4 & -2 \\ 0 & -3 & 2 & 4 \\ 0 & 6 & -4 & -8 \end{vmatrix} r_1 \leftrightarrow r_2 = (-1) \begin{vmatrix} 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 0 & -3 & 2 & 4 \\ 0 & 6 & -4 & -8 \end{vmatrix}$$

$$r_4 + 2r_3 \to r_4 = (-1) \begin{vmatrix} 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 0 & -3 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Notice that since we encountered a row of zeros, then $det(A) = (-1)(0) = 0$.

19. If $det(AB) = 0$, must $det(A)$ or $det(B) = 0$? Justify your answer.

Suppose that $det(AB) = 0$. Since $det(AB) = det(A)det(B)$, then we have $det(A)det(B) = 0$. Recall that a product of two real numbers is zero if and only if at least one of the numbers is zero. Thus we must have either $det(A) = 0$ or $det(B) = 0$ (including the possibility that both of them are zero).

20. Show that if k is a scalar and A is an $n \times n$ matrix, then $det(kA) = k^n det(A)$.

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let $B = [ka_{ij}]$ be the matrix obtained by multiplying A by a constant k. If $k = 0$ then B has a row of zeros and hence $det(B) = 0 = 0^n det(A)$. Otherwise, using the definition, $det(B) = \sum(\pm) b_{1j_1} b_{2j_2} \cdots b_{\ell j_{ell}} \cdots b_{nj_n} = \sum(\pm) ka_{1j_1} ka_{2j_2} \cdots ka_{\ell j_{ell}} \cdots ka_{nj_n}$ (since the entries have all been scaled by k)
Then, factoring out the k from each term and pulling it outside the sum, $det(B) = k^n \sum(\pm) a_{1j_1} a_{2j_2} \cdots a_{\ell j_{ell}} \cdots a_{nj_n} = k^n det(A)$. □

21. Prove Theorem 3.5

Theorem 3.5: If B is obtained from A by multiplying a row (column) of A by a real number k, then $det(B) = kdet(A)$.

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let $B = [b_{ij}]$ be the matrix obtained by multiplying the ℓth row of A by a constant k. If $k = 0$ then B has a row of zeros and hence $det(B) = 0 = 0det(A)$. Otherwise, using the definition, $det(B) = \sum(\pm) b_{1j_1} b_{2j_2} \cdots b_{\ell j_{ell}} \cdots b_{nj_n} = \sum(\pm) a_{1j_1} a_{2j_2} \cdots ka_{\ell j_{ell}} \cdots a_{nj_n}$ (since the entries for all other rows remain the same, and the entry drawn from row ℓ has been scaled by k)
Then, factoring out the k from each term and pulling it outside the sum, $det(B) = k \sum(\pm) a_{1j_1} a_{2j_2} \cdots a_{\ell j_{ell}} \cdots a_{nj_n} = kdet(A)$

The column case follows by observing that a column operation on A is a row operation on A^T and using the fact that $det(A) = det(A^T)$. □.

22. Prove Theorem 3.8

Theorem 3.8: If A is an $n \times n$ matrix, then A is nonsingular if and only if $det(A) \neq 0$.

Proof:

First, suppose that A is nonsingular. Then, by Theorem 2.8, $A = E_1 E_2 \cdots E_n$ where each E_i is an elementary matrix. Then, using Lemma 3.1 n times, $det(A) = det(E_1 E_2 \cdots E_n) = det(E_1)det(E_2)\cdots det(E_n) \neq 0$ (as discussed in the sketch of the proof of Lemma 3.1 above, for each i, $det(E_i) \neq 0$).

Conversely, suppose A is singular. Then, using Theorem 2.10, A is row equivalent to a matrix B with a row of zeros. Then, by Theorem 3.4, $det(B) = 0$. Therefore, since $A = F_1 F_2 \cdots F_\ell B$, where each F_i is an elementary matrix, $det(A) = det(F_1 F_2 \cdots F_\ell B) = det(F_1)det(F_2)\cdots det(F_\ell)det(B) = 0$.

23. Let $A = \begin{bmatrix} 3 & 1 & 0 & -1 \\ 2 & 0 & -5 & 1 \\ 2 & 1 & -4 & 0 \\ 0 & -1 & 3 & 2 \end{bmatrix}$. Find:

(a) $M_{14} = \begin{bmatrix} 2 & 0 & -5 \\ 2 & 1 & -4 \\ 0 & -1 & 3 \end{bmatrix}$

(b) $M_{32} = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 1 \\ 0 & 3 & 2 \end{bmatrix}$

(c) $det(M_{32}) = \begin{vmatrix} 3 & 0 & -1 \\ 2 & -5 & 1 \\ 0 & 3 & 2 \end{vmatrix} = (-30) + (0) + (-6) - (0) - (0) - (9) = -45$

(d) $A_{32}(-1)^{3+2}det(M_{32}) = (-1)(-45) = 45$

(e) $A_{41} = (-1)^{4+1}det(M_{41}) = (-1)\begin{vmatrix} 1 & 0 & -1 \\ 0 & -5 & 1 \\ 1 & -4 & 0 \end{vmatrix} = (0) + (0) + (0) - (5) - (0) - (-4) = (-1)(-1) = 1$

24. Find the determinant of the matrix A from the previous problem by using the expansion of $det(A)$ along the last row.

$$det(A) = (0)A_{41} + (-1)A_{42} + (3)A_{43} + (2)A_{44} = 0 + (-1)\begin{vmatrix} 3 & 0 & -1 \\ 2 & -5 & 1 \\ 2 & -4 & 0 \end{vmatrix} + (3)\begin{vmatrix} 3 & 1 & -1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} + (2)\begin{vmatrix} 3 & 1 & 0 \\ 2 & 0 & -5 \\ 2 & 1 & -4 \end{vmatrix}$$

$$= 0+(-1)[(0)+(0)+(8)-(10)-(0)-(-12)]+(3)[(0)+(2)+(-2)-(0)-(0)-(-3)]+(2)[(0)+(-10)+(0)-(0)-(-8)-(-15)]$$

$$= 0 + (-1)(10) + (3)(-3) + (2)(13) = 0 - 10 - 9 + 26 = 7$$