# MAT 2384 3X Assignment #1 Solutions

1. $y' = \dfrac{1+y^2}{x^2+4}$, $y(0) = 1$

this equation is obviously separable : $\dfrac{dy}{1+y^2} = \dfrac{dx}{4+x^2}$

integrate on both sides $\displaystyle\int \frac{dy}{1+y^2} = \int \frac{dx}{4+x^2} + C$

to get $\arctan(y) = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$

or

$$\boxed{y = \tan\left(\tfrac{1}{2}\arctan\left(\tfrac{x}{2}\right) + C\right)}$$ (general solution)

then $y(0) = 1 \Rightarrow 1 = \tan(0 + C) \Rightarrow C = \pi/4$

so the unique solution is $\boxed{y = \tan\left(\frac{1}{2}\arctan\left(\frac{x}{2}\right) + \frac{\pi}{4}\right)}$

2. $(x+2y)\,dx - x\,dy = 0$, $y(1) = 3$ (not separable)

$M(x,y) = x + 2y$ $\quad M_y = 2$
$N(x,y) = -x$ $\quad N_x = -1$

$M_y \neq N_x$, so DE is not exact

$M_y - N_x = 2 - (-1) = 3$ then $\dfrac{M_y - N_x}{N} = \dfrac{3}{-x}$ (a function of $x$ only)

so $\mu(x) = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$ is the integrating factor

multiplying the DE by $\mu(x)$, we get $(x^{-2} + 2x^{-3}y)\,dx - x^{-2}\,dy = 0$

hence $M^*(x,y) = x^{-2} + 2x^{-3}y$ $\quad M^*_y = 2x^{-3}$
$N^*(x,y) = -x^{-2}$ $\quad N^*_x = 2x^{-3}$

$M^*_y = N^*_x$ so the DE is now exact

$F(x,y) = \int M^*(x,y)\,dx + g(y)$ $\quad \left(\text{or } \int N^*(x,y)\,dy + h(x)\right)$

(continued →)

$$F(x,y) = \int (x^{-2} + 2x^{-3}y)\,dx + g(y) = -x^{-1} - x^{-2}y + g(y)$$

then $\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(-x^{-1} - x^{-2}y + g(y)\right) = -x^{-2} + g'(y) = N^*(x,y) = -x^{-2}$

and so $g'(y) = 0$, then $g(y) = $ constant, so take $g(y) = 0$

$\therefore F(x,y) = -x^{-1} - x^{-2}y$

and the general solution is $-x^{-1} - x^{-2}y = C$ or $\boxed{y = -x - Cx^2}$

$y(1) = 3 \Rightarrow 3 = -1 - C \Rightarrow C = -4$

$\therefore$ the unique solution is $\boxed{y = 4x^2 - x}$

OR since $M(x,y)$ and $N(x,y)$ are both homogeneous of degree 1, we could use the substitution $y = ux$, $dy = u\,dx + x\,du$
(or the substitution $x = uy$, $dx = u\,dy + y\,du$)

the DE becomes $(x + 2ux)dx - x(u\,dx + x\,du) = 0$
which is $x\,dx + 2ux\,dx - ux\,dx - x^2 du = 0$
or $x\,dx + ux\,dx - x^2 du = 0$
or $x(1+u)dx - x^2 du = 0$
which separates as $\frac{du}{1+u} = \frac{dx}{x}$

integrate on both sides $\int \frac{du}{1+u} = \int \frac{dx}{x} + C$ to get

$$\ln|1+u| = \ln|x| + C$$

exponentiate both sides to get $1 + u = Kx$ or $u = Kx - 1$

or $\underline{y = Kx^2 - x}$

3. $(\cos y + y^2\cos x + 1)\,dx + (2y\sin x - x\sin y)\,dy = 0$, $y(\pi) = \pi$ (not separable)

$M(x,y) = \cos y + y^2\cos x + 1$ $\quad M_y = -\sin y + 2y\cos x$ $\quad M_y = N_x$

$N(x,y) = 2y\sin x - x\sin y$ $\quad N_x = 2y\cos x - \sin y$ $\quad \therefore$ DE is exact

$$F(x,y) = \int N(x,y)\,dy + h(x) \qquad \left(\text{or } \int M(x,y)\,dx + g(y)\right)$$

$$= \int (2y\sin x - x\sin y)\,dy + h(x) = y^2\sin x + x\cos y + h(x)$$

then $\dfrac{dF}{dx} = \dfrac{d}{dx}\left(y^2\sin x + x\cos y + h(x)\right) = y^2\cos x + \cos y + h'(x) = M(x,y)$
$= \cos y + y^2\cos x + 1$

so $h'(x) = 1 \Rightarrow h(x) = x$

$\therefore F(x,y) = y^2\sin x + x\cos y + x$

and the general solution is $\boxed{y^2\sin x + x\cos y + x = C}$

$y(\pi) = \pi \Rightarrow (\pi)^2\sin(\pi) + (\pi)\cos(\pi) + \pi = C \Rightarrow C = 0$

$\therefore$ the unique solution is $\boxed{y^2\sin x + x\cos y + x = 0}$

4. $(x + xy^2)\,dx + y\,dy = 0$, $y(0) = 2$

this equation is separable $\quad x(1+y^2)\,dx + y\,dy = 0$

separates as $\quad \dfrac{y\,dy}{1+y^2} = -x\,dx$

integrate on both sides $\quad \displaystyle\int \frac{y\,dy}{1+y^2} = \int -x\,dx + C$

to get $\quad \frac{1}{2}\ln(1+y^2) = -\frac{1}{2}x^2 + C$

or $\quad \ln(1+y^2) = C - x^2$

(continued →)

or $\quad 1+y^2 = Ke^{-x^2}$ $\quad$ or $\quad \boxed{y^2 = Ke^{-x^2} - 1}$ (general solution)

$y(0)=2 \Rightarrow 4 = Ke^0 - 1 \Rightarrow K = 5$ (and $y>0$)

$\therefore$ the unique solution is $\quad \boxed{y = \sqrt{5e^{-x^2}-1}}$

OR $\quad M(x,y) = x + xy^2 \qquad M_y = 2xy \qquad M_y \neq N_x$ so the DE
$\quad N(x,y) = y \qquad N_x = 0 \qquad$ is not exact

$M_y - N_x = 2xy$ so $\dfrac{M_y - N_x}{N} = \dfrac{2xy}{y} = 2x$ (a function of $x$ only)

and the integrating factor is $\mu(x) = e^{\int 2x\,dx} = e^{x^2}$

then the DE becomes $\quad e^{x^2}(x + xy^2)dx + e^{x^2}y\,dy = 0$

$M^*(x,y) = e^{x^2}(x+xy^2) \qquad M^*_y = 2xye^{x^2} \qquad M^*_y = N^*_x$
$N^*(x,y) = ye^{x^2} \qquad N^*_x = 2xye^{x^2} \qquad$ so DE is now exact

$F(x,y) = \int N^*(x,y)dy + h(x) = \int e^{x^2}y\,dy + h(x) = \frac{1}{2}y^2e^{x^2} + h(x)$

then $\dfrac{\partial F}{\partial x} = \dfrac{\partial}{\partial x}\left(\frac{1}{2}y^2e^{x^2} + h(x)\right) = xy^2e^{x^2} + h'(x) = M^*(x,y) = e^{x^2}(x+xy^2)$

so $\quad h'(x) = xe^{x^2} \Rightarrow h(x) = \frac{1}{2}e^{x^2}$

$\therefore F(x,y) = \frac{1}{2}y^2e^{x^2} + \frac{1}{2}e^{x^2} = \frac{1}{2}e^{x^2}(1+y^2)$

so the general solution is $\quad e^{x^2}(1+y^2) = C \quad$ or $\quad \underline{y^2 = Ce^{-x^2} - 1}$

5. $(y\cos(x+y))\,dx + (3\sin(x+y) + y\cos(x+y))\,dy = 0, \quad y(0) = \pi/2$ (not separable)

$M(x,y) = y\cos(x+y) \qquad M_y = \cos(x+y) - y\sin(x+y) \qquad M_y \neq N_x$
$N(x,y) = 3\sin(x+y) + y\cos(x+y) \qquad N_x = 3\cos(x+y) - y\sin(x+y)$ so DE not exact

$$M_y - N_x = \cos(x+y) - y\sin(x+y) - (3\cos(x+y) - y\sin(x+y)) = -2\cos(x+y)$$

then $\dfrac{M_y - N_x}{M} = \dfrac{-2\cos(x+y)}{y\cos(x+y)} = -\dfrac{2}{y}$ (a function of $y$ only)

the integrating factor is $\mu(y) = e^{-\int -\frac{2}{y}dy} = e^{2\ln y} = y^2$

and the DE becomes $y^3\cos(x+y)\,dx + (3y^2\sin(x+y) + y^3\cos(x+y))\,dy = 0$

$M^*(x,y) = y^3\cos(x+y) \qquad M^*_y = 3y^2\cos(x+y) - y^3\sin(x+y) \qquad M^*_y = N^*_x$
$N^*(x,y) = 3y^2\sin(x+y) + y^3\cos(x+y) \qquad N^*_x = 3y^2\cos(x+y) - y^3\sin(x+y)$ $\therefore$ DE exact

$$F(x,y) = \int M^*(x,y)\,dx + g(y) = \int y^3\cos(x+y)\,dx + g(y) = y^3\sin(x+y) + g(y)$$

then $\dfrac{\partial F}{\partial y} = \dfrac{\partial}{\partial y}\left(y^3\sin(x+y) + g(y)\right) = 3y^2\sin(x+y) + y^3\cos(x+y) + g'(y)$
$= N^*(x,y) = 3y^2\sin(x+y) + y^3\cos(x+y)$
$\Rightarrow g'(y) = 0 \Rightarrow g(y) = K$, take $g(y) = 0$

$\therefore F(x,y) = y^3\sin(x+y)$ and the general solution is $\boxed{y^3\sin(x+y) = C}$

$y(0) = \pi/2 \Rightarrow \left(\frac{\pi}{2}\right)^3 \sin(0 + \pi/2) = C \Rightarrow C = \pi^3/8$

and the unique solution is $\boxed{y^3\sin(x+y) = \dfrac{\pi^3}{8}}$

6. $(x+1)\,dx + (y+3)\,dy = 0$ , $y(1) = 4$

this equation is separable $(y+3)\,dy = -(x+1)\,dx$

integrate on both sides $\int (y+3)\,dy = -\int (x+1)\,dx + C$

to get the general solution $\frac{1}{2}y^2 + 3y = -\frac{1}{2}x^2 - x + C$

or $\frac{1}{2}y^2 + 3y + \frac{1}{2}x^2 + x = C$ or $\boxed{y^2 + 6y + x^2 + 2x = C}$

$y(1) = 4 \Rightarrow (4)^2 + 6(4) + (1)^2 + 2(1) = C \Rightarrow C = 43$

$\therefore$ the unique solution is $\boxed{y^2 + 6y + x^2 + 2x = 43}$

OR $M(x,y) = x+1$ $M_y = 0$ and so $M_y = N_x$ and

$N(x,y) = y+3$ $N_x = 0$ the DE is exact

then $F(x,y) = \int N(x,y)\,dy + h(x)$ (or $\int M(x,y)\,dx + g(y)$)

$= \int (y+3)\,dy + h(x) = \frac{1}{2}y^2 + 3y + h(x)$

so $\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2}y^2 + 3y + h(x)\right) = h'(x) = M(x,y) = x+1 \Rightarrow h(x) = \frac{1}{2}x^2 + x$

and then $F(x,y) = \frac{1}{2}y^2 + 3y + \frac{1}{2}x^2 + x$ and the general solution is

$\underline{\frac{1}{2}y^2 + 3y + \frac{1}{2}x^2 + x = C}$

7. $(\cos x - 2x\sin x - 2y\sin x)dx + \cos x\,dy = 0$ , $y(0) = 5$ (not separable)

$M(x,y) = \cos x - 2x\sin x - 2y\sin x$ $\quad M_y = -2\sin x$ $\quad M_y \neq N_x$, so
$N(x,y) = \cos x$ $\quad N_x = -\sin x$ $\quad$ the DE is not exact

$M_y - N_x = -2\sin x - (-\sin x) = -\sin x$

then $\dfrac{M_y - N_x}{N} = \dfrac{-\sin x}{\cos x}$ (a function of $x$ only)

and the integrating factor is $\mu(x) = e^{\int -\sin x/\cos x\,dx} = e^{\ln\cos x} = \cos x$

and the DE becomes $(\cos^2 x - 2x\cos x\sin x - 2y\cos x\sin x)dx + \cos^2 x\,dy = 0$

$M^*(x,y) = \cos^2 x - 2x\cos x\sin x - 2y\cos x\sin x$ $\quad M^*_y = -2\cos x\sin x$
$N^*(x,y) = \cos^2 x$ $\quad N^*_x = -2\cos x\sin x$

$M^*_y = N^*_x$ and so the DE is now exact

$F(x,y) = \int N^*(x,y)dy + h(x) = \int \cos^2 x\,dy + h(x) = y\cos^2 x + h(x)$

so $\dfrac{\partial F}{\partial x} = \dfrac{\partial}{\partial x}\left(y\cos^2 x + h(x)\right) = -2y\cos x\sin x + h'(x) = M^*(x,y)$
$= \cos^2 x - 2x\cos x\sin x - 2y\cos x\sin x$

so $h'(x) = \cos^2 x - 2x\cos x\sin x$

then $h(x) = \int(\cos^2 x - 2x\cos x\sin x)dx = x\cos^2 x$

so $F(x,y) = (x+y)\cos^2 x$ and the <u>general solution</u> is $(x+y)\cos^2 x = C$

<u>or</u> $\boxed{y = C\sec^2 x - x}$

$y(0) = 5 \Rightarrow 5 = C\sec^2(0) - 0 \Rightarrow C = 5$

$\therefore$ the <u>unique solution</u> is $\boxed{y = 5\sec^2 x - x}$

8. want $x$ such that $f(x) = x^3 + 7x - 6 = 0$

so $7x = 6 - x^3$ or $x = \dfrac{6 - x^3}{7}$

$\therefore$ take $g(x) = \dfrac{6 - x^3}{7}$

then $|g'(x)| = \left| -\dfrac{3x^2}{7} \right| = \dfrac{3}{7} x^2 \leq \dfrac{3}{7}$ on $[0,1]$

$\therefore$ the sequence generated by $x_{n+1} = g(x_n)$ will converge

$x_0 = 0.75 \qquad x_1 = g(x_0) = \dfrac{6 - x_0^3}{7} = \dfrac{6 - (0.75)^3}{7} = 0.79688$

$x_2 = g(x_1) = \dfrac{6 - (0.79688)^3}{7} = 0.78485$

$x_3 = g(x_2) = \dfrac{6 - (0.78485)^3}{7} = 0.78808$

$x_4 = g(x_3) = \dfrac{6 - (0.78808)^3}{7} = 0.78722$

$x_5 = g(x_4) = \dfrac{6 - (0.78722)^3}{7} = 0.78745$

$x_6 = g(x_5) = \dfrac{6 - (0.78745)^3}{7} = 0.78739$

$x_7 = g(x_6) = \dfrac{6 - (0.78739)^3}{7} = 0.78740$

$x_8 = g(x_7) = \dfrac{6 - (0.78740)^3}{7} = 0.78740 \quad \therefore$ stop

$\therefore$ the root is $\underline{\underline{0.78740}}$ to 5 decimal places

( check : $f(0.78740) = (0.78740)^3 + 7(0.78740) - 6 \simeq -1.3 \times 10^{-5}$ )

9. To solve $x = \cos x$, take $f(x) = x - \cos x$

then $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n} = \frac{x_n + x_n \sin x_n - x_n + \cos x_n}{1 + \sin x_n}$

$= \frac{x_n \sin x_n + \cos x_n}{1 + \sin x_n}$

$x_0 = \pi/4$, $\quad x_1 = g(x_0) = \frac{(x_0)\sin(x_0) + \cos(x_0)}{1 + \sin(x_0)}$

$= \frac{(\pi/4)\sin(\pi/4) + \cos(\pi/4)}{1 + \sin(\pi/4)} \simeq 0.739536$

$x_2 = \frac{(0.739536)\sin(0.739536) + \cos(0.739536)}{1 + \sin(0.739536)} \simeq 0.739085$

$x_3 = \frac{(0.739085)\sin(0.739085) + \cos(0.739085)}{1 + \sin(0.739085)} \simeq 0.739085$

$\therefore$ stop

$\therefore$ the solution is $\underline{x \approx 0.739085}$