DEPARTMENT OF MATHEMATICS
MID TERM TEST 1 SOLUTIONS
MTH 141 – LINEAR ALGEBRA
Date: October 17th, 2015, 9:10 am
Duration: 1.5 hours

Last Name:____________________________________

First Name:___________________________________

Ryerson email: ________________________________

Student Number(ONLY LAST 3 DIGITS): ___________

Signature:____________________________________

**Section (circle one)**

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|---|---|---|---|---|---|
| A. Alvarez : | 1 | 2 | 3 | 4 | 25 |
| M.Alqasas : | 5 | 6 | 7 | 8 | 9 |
| N. Jung : | 10 | 11 | 12 | 13 | 14 |
| C. Wang : | 15 | 16 | 17 | 18 | 19 |
| L. Fisseha : | 20 | 21 | 22 | 23 | 24 |

**Instructions:**

1. This is a closed-book test. **Notes, calculators and other aids are not permitted.**
2. Verify that your test has pages 1-10. The exam questions are in pages 2-8. Pages 9 and 10 are given for extra space, they do NOT contain any questions.
3. (a) Unless otherwise instructed, **make sure you include all significant steps in your solution, presented in the correct order. Unjustified answers will be given little or no credit. Cross out or erase all rough work not relevant to your solution.** Put a box around your final **answer.**
   (b) For multiple choice questions make sure to write your answers in the box at the end of each question **carefully**. There are no part marks in the multiple-choice section and **only** the answer in the box will be marked. The correct response gets full marks, an incorrect response or no response gets no marks.
   (c) Write your solutions in the space provided. **ANYTHING WRITTEN IN THE BACK OF ANY PAGE WILL NOT BE MARKED**. If you need more space, use the front part of page 9 and page 10. Indicate this fact on the original page.
4. Do not separate the sheets.
5. Have your student card available on your desk.

1. [2 marks] (Multiple choice question). Let $z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $z_2 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$.

   Then $\frac{z_1}{z_2}$ is equal to:

   A) $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ B) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ C) $-\sqrt{3} + i$ D) $\sqrt{3} + i$ E) None of these

   Write the (capital) letter of the answer in this box. **Only** the answer in the box will be marked. | D ) |

2. [2 marks] (Multiple choice question). Consider the vector $\vec{x} = \begin{bmatrix} 2 \\ -2 \\ 4 \\ 1 \end{bmatrix}$. A unit vector in the direction of $\vec{x}$ is

   A) $\vec{y} = \begin{bmatrix} 2/7 \\ -2/7 \\ 4/7 \\ 1/7 \end{bmatrix}$ B) $\vec{y} = \begin{bmatrix} 2/5 \\ -2/5 \\ 4/5 \\ 1/5 \end{bmatrix}$ C) $\vec{y} = \begin{bmatrix} 2/25 \\ -2/25 \\ 4/25 \\ 1/25 \end{bmatrix}$ D) None of these

   Write the (capital) letter of the answer in this box. **Only** the answer in the box will be marked. | B ) |

3. [2 marks] (Multiple choice question). The vectors $\begin{bmatrix} -1 \\ k \\ -2 \end{bmatrix}$ and $\begin{bmatrix} k \\ 3 \\ 1 \end{bmatrix}$ are orthogonal if:

   A) $k = 0$ B) $k = -1$ C) $k = 5$ D) $k = -2$ E) None of these

   Write the (capital) letter of the answer in this box. **Only** the answer in the box will be marked. | E ) |

4. [7 marks] Use polar form to determine all the indicated roots: $(-8+8\sqrt{3}i)^{1/4}$. Express these roots in standard form.

**Solution:**

If $Z = -8+8\sqrt{3}i$, then $|Z| = \sqrt{64+3\cdot 64} = \sqrt{4\cdot 64} = 2\cdot 8 = 16$. An argument for $Z$ is $\theta = 2\pi/3$. So, in polar form $Z$ can be written as $Z = 16e^{i\cdot\frac{2\pi}{3}}$.

The fourth complex roots of $Z$ are given by the formula $Z_k = |Z|^{1/4}e^{i\cdot\frac{\theta+2k\pi}{4}}$, with $k = 0, 1, 2, 3$.

Then we have:

$Z_0 = 2e^{i\cdot\frac{\pi}{6}}$
$Z_1 = 2e^{i\cdot\left(\frac{\pi}{6}+\frac{\pi}{2}\right)} = 2e^{i\frac{2\pi}{3}}$
$Z_2 = 2e^{i\cdot\left(\frac{\pi}{6}+\pi\right)} = 2e^{i\frac{7\pi}{6}}$
$Z_3 = 2e^{i\cdot\left(\frac{\pi}{6}+\frac{3\pi}{2}\right)} = 2e^{i\frac{5\pi}{3}}$

In standard form, these roots are expressed as:

$Z_0 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i$

$Z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1 + i\sqrt{3}$

$Z_2 = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 2\left(\frac{-\sqrt{3}}{2} - i\frac{1}{2}\right) = -\sqrt{3} - i$

$Z_1 = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}$

5. [7 marks] Consider the following set of $\mathbb{R}^4$.

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| \, x_1x_2 = x_3x_4 \right\}$$

a) Is the set $S$ closed under addition? Explain.
b) Is the set $S$ closed under scalar multiplication? Explain.
c) Is the set $S$ a subspace of $\mathbb{R}^4$? Explain.

**Solution:**

a) The set $S$ is not close under addition. Take for example $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Both $\vec{x}$ and $\vec{y}$ belong to $S$, but $\vec{x} + \vec{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ does not belong to $S$.

b) Assume that $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ belong to $S$. Then $x_1x_2 = x_3x_4$. Let $t$ be any arbitrary real number. Then $t\vec{x} = \begin{bmatrix} tx_1 \\ tx_2 \\ tx_3 \\ tx_4 \end{bmatrix}$ and $(tx_1)(tx_2) = t^2x_1x_2 = t^2x_3x_4 = (tx_3)(tx_4)$. Therefore $t\vec{x}$ belongs to $S$ and we can conclude that $S$ is closed under scalar multiplication.

c) The set $S$ is NOT a subspace because $S$ is not closed under addition.

6. [6 marks] Use cross product to determine a vector equation of the line of intersection of the planes:

$$2x_1 + 3x_2 - 4x_3 = 1$$
$$x_1 - 2x_2 - x_3 = -4$$

**Solution:** To find a vector equation of the line of intersection of the form $\vec{x} = \vec{p} + t\vec{d}$ we need a point $\vec{p}$ that belongs to the line and the direction vector $\vec{d}$ of the line.

A normal vector to the plane $2x_1 + 3x_2 - 4x_3 = 1$ is $\vec{n}_1 = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and a normal vector to the plane $x_1 - 2x_2 - x_3 = -4$ is $\vec{n}_2 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

Then, the line of intersection of the two planes has direction

$$\vec{d} = \vec{n}_1 \times \vec{n}_2 = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -11 \\ -2 \\ -7 \end{bmatrix}$$

To find a point in the line, we can make $x_3 = 0$ and solve the system

$$2x_1 + 3x_2 = 1$$
$$x_1 - 2x_2 = -4$$

The solution to that system of equations is $x_2 = 9/7$ and $x_1 = -10/7$.

Then a vector equation of the line of intersection is

$$\vec{x} = \begin{bmatrix} -\frac{10}{7} \\ \frac{9}{7} \\ 0 \end{bmatrix} + t \begin{bmatrix} -11 \\ -2 \\ -7 \end{bmatrix}$$

**Remark:** When finding $\vec{d}$ it is also correct to consider the cross product $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$.

Additionally, there are many ways to find a point $\vec{p}$ that belongs to the line. For example, instead of taking $x_3 = 0$ another possibility could be to take $x_2 = 0$.

7. [8 marks] Find the distance from the point $Q(2,3,2)$ to the line

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

**Solution:** Take the point $P(1,1,-1)$ that belongs to the line. Then, the distance from point $Q$ to the line is $\left\| perp_{\vec{d}} \vec{PQ} \right\|$, where $\vec{d} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ is the direction vector of the line, and

$$\vec{PQ} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Then we have

$$\left\| perp_{\vec{d}} \vec{PQ} \right\| = \left\| \vec{PQ} - proj_{\vec{d}} \vec{PQ} \right\| = \left\| \vec{PQ} - \frac{\vec{d} \cdot \vec{PQ}}{\|\vec{d}\|^2} \vec{d} \right\| = \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{12}{18} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 1/3 \\ -2/3 \\ 7/3 \end{bmatrix} \right\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{49}{9}} = \sqrt{\frac{54}{9}} = \sqrt{6}$$

8. [7 marks] Let $\vec{u} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$

a) Find the length of vector $\vec{u}$
b) Are vectors $\vec{u}$ and $\vec{v}$ orthogonal? Explain.
c) Find the area of the parallelogram determined by vectors $\vec{v}$ and $\vec{w}$

**Solution:**

a) The length of vector $\vec{u}$ is $\|\vec{u}\| = \left\| \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \right\| = \sqrt{1+16+4} = \sqrt{21}$

b) $\vec{u} \cdot \vec{v} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} = -3+8-4 = 1 \neq 0$. Therefore the vectors $\vec{u}$ and $\vec{v}$ are NOT orthogonal.

c) The area of the parallelogram determined by vectors $\vec{v}$ and $\vec{w}$ is:

$$A = \|\vec{v} \times \vec{w}\| = \left\| \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} \times \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 4 \\ 5 \\ 11 \end{bmatrix} \right\| = \sqrt{162}$$

9. [9 marks] For the following system of linear equations

$$\begin{aligned} x_1 - x_2 + 2x_3 - 3x_4 &= -4 \\ 3x_1 - 3x_2 + 8x_3 - 5x_4 &= -4 \\ 2x_1 - 2x_2 + 5x_3 - 4x_4 &= -4 \\ 3x_1 - 3x_2 + 7x_3 - 7x_4 &= -8 \end{aligned}$$

i) Write the augmented matrix
ii) Obtain a row equivalent matrix in reduced row echelon form (RREF)
iii) Determine whether the system is consistent or inconsistent
iv) What is the rank of the augmented matrix? Explain
v) If the system is consistent, find the general solution.

**Solution:**

i) and ii) $\left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & -4 \\ 3 & -3 & 8 & -5 & -4 \\ 2 & -2 & 5 & -4 & -4 \\ 3 & -3 & 7 & -7 & -8 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & -4 \\ 0 & 0 & 2 & 4 & 8 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right] \sim$

$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & -4 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & -7 & -12 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

iii) the system is consistent

iv) the rank of the augmented matrix is 2, because the RREF of the matrix has two leading ones.

v) take $x_4 = t$, $x_2 = s$, then we have:

$x_3 = 4 - 2t$, $x_1 = -12 + 7t + s$

Solution: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -12 + 7t + s \\ s \\ 4 - 2t \\ t \end{bmatrix} = \begin{bmatrix} -12 \\ 0 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ 0 \\ -2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$