Concordia University
STAT 250/2AA
Fall 2010

Midterm Exam -Solutions

1. Let Y_1, Y_2 be random variables with joint density function

$$f(y_1, y_2) = y_1 e^{-y_1 y_2} \text{ for } 0 \leq y_1 \leq 1, y_2 \geq 0$$
$$= 0 \text{ otherwise}$$

Calculate:

(a) (8 pts) Find the joint density function of the random variables $U_1 = Y_1$ and $U_2 = Y_1Y_2$.
– you can use the Method of Transformations to get that

$$f(u_1, u_2) = e^{-u_2}, 0 \leq u_1 \leq 1, u_2 \geq 0, \text{ and } f(u_1, u_2) = 0 \text{ otherwise.}$$

(b) (4 pts) Find the marginal density of U_2 and identify whether it is a well known type of random variable.
– you need to integrate result in (a) over $0 \leq u_1 \leq 1$ to find that U_2 is Exponential ($\beta = 1$)

(c) (4 pts) Find $E(U_1)$ and $V(U_1)$
– if you integrate result in (a) over $u_2 \geq 0$ you will get that U_1 is Uniform([0, 1]) so

$$E(U_1) = 1/2, \ V(U_1) = 1/12$$

(d) (4 pts) Are U_1 and U_2 independent? You must prove your answer.
– yes, because results from (b) and (c) show that $f(u_1, u_2) = f_{U_1}(u_1) f_{U_2}(u_2)$ holds exactly.

2. (10 pts) Let Y_1 and Y_2 have a joint density that is uniform over $(y_1, y_2) : y_1 \in (0, 1), y_2 \in (0, 2)$. Find the density of the variable $U = 2Y_1 + Y_2$.
– you can use the Method of Distributions to show that $U \in (0, 4)$ and

$$P(U \leq u) = u^2/8 \text{ when } 0 < u \leq 2; \text{ and } P(U \leq u) = 1 - (4-u)^2/8 \text{ when } 2 \leq u < 4$$

then differentiating this with respect to u we get that the density is

$$f(u) = u/4, 0 < u < 4, \text{ and } f(u) = 0 \text{ otherwise.}$$

3. Let there be independent trials of independent experiments all with the same probability of success p that is unknown. Suppose that this unknown probability of success $p = X$ is a random variable with density

$$f(p) = 3p^2 \text{ for } 0 \leq p \leq 1$$
$$= 0 \text{ otherwise}$$

. Let N be an the number of the trial on which the first success occurs.
– we have that conditional distribution of N given p is Geometric(p), so

$$E[N|p] = 1/p,\ V[N|p] = (1-p)/p^2$$

– the density of the variable p is used to compute $E(g(p)), V(g(p))$ for different functions $g(p)$ below:

(a) (5 pts) Find $E(N)$.

$$E(N) = E(E[N|p]) = E(1/p) = 3/2$$

(b) (8 pts) Find $V(N)$.

$$V(N) = V(E[N|p]) + E(V[N|p]) = V(1/p) + E((1-p)/p^2) = [3-(3/2)^2] + [3-(3/2)] = 9/4$$

4. Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with the same density $f(y) = \lambda e^{-\lambda y}, y > 0$, and let $U = Y_1 + Y_2 + \cdots + Y_n$.

   (a) (8 pts) What random variable is U. You must justify your answer with a proof.
   – you can use the Method of Moment Generating Functions to show that

   $$m_U(t) = (1 - t/\lambda)^{-n}$$

   – so U is a Gammma($\beta = 1/\lambda, \alpha = n$)

   (b) (4 pts) What are the values for $E(U)$ and $V(U)$?
   – you can use result in (a) or a direct calculation from $U = Y_1 + Y_2 + \cdots + Y_n$ to get

   $$E(U) = n/\lambda,\ V(U) = n/\lambda^2$$

5. (8 pts) Let X, Y be independent random variables with Uniform distribution on $[0,1]$. Let $U = \min(X,Y)$. Find the density of the random variable U and calculate $E(U)$.
– you can use $P(U \le u) = 1 - P(X > u, Y > u)$ and the independence of X, Y to get that

$$f(u) = 2(1-u),\ 0 < u < 1;\ \text{and}\ \ f(u) = 0\ \ \text{otherwise}$$

– and then calculate that $E(U) = 1/3$

6. A sample of prices for the same item over 100 different stores has been collected. The mean price of this item is unknown. The standard deviation of the price is known to be equal to 10.
– μ is unknown, $\sigma^2 = (10)^2$, $n = 100$

   (a) (8 pts) What is the approximate probability that the sample mean of the prices over these 100 stores will be within 1.5 of the true mean price?

   $$P(|\bar{X} - \mu| \le 1.5) = P(Z \le 1.5) = 0.664$$

(b) (7 pts) If we want the sample mean to be within 1.5 of the true mean price with approximate probability 0.95, how many stores should we sample the prices from?
– here you do not use above value of n, instead you solve for n the equation:

$$0.025 = P(|\bar{X} - \mu| \leq 1.5)] = P(Z \leq \sqrt{n}1.5/10)$$

– to get that $n = 170.78$ so you need at least 171 stores

(c) (2 pts) Why did we say that the probabilities above are 'approximate' (we did not say anything about the distribution of the prices in the population)?
– because we did not assume the process were Normal(μ, σ^2) and instead since $n = 100$ is large
– we used the Central Limit Theorem to say that $\bar{X}$ is approximately Normal$(\mu, \sigma^2/\sqrt{n})$