Midterm Exam II ENGR 213
Applied Ordinary Differential Equations

Fall 2014
Solutions

Problem 1. Given the equation

$$y^{(4)} + y'' = 0,$$

verify that the functions

$$y_1(x) = 1, \quad y_2(x) = x, \quad y_3(x) = \cos x, \quad y_4(x) = \sin x$$

form a fundamental set of solutions on the interval $(-\infty, \infty)$. If so, write the general solution of the given equation.

Solution.

1). Verify that the given functions satisfy the differential equation.

(a) Since $y_1''(x) = y_1^{(4)}(x) \equiv 0$, $y_1(x)$ satisfies the differential equation.

(b) Since $y_2''(x) = y_2^{(4)}(x) \equiv 0$, $y_2(x)$ satisfies the differential equation.

(c) Since $y_3''(x) = -\cos x$, $y_3^{(4)}(x) = \cos x$, and $y_3^{(4)}(x) - y_3''(x) = \cos x - \cos x = 0$, $y_3(x)$ satisfies the differential equation.

(d) Since $y_4''(x) = -\sin x$, $y_4^{(4)}(x) = \sin x$, and $y_4^{(4)}(x) - y_4''(x) = \sin x - \sin x = 0$, $y_4(x)$ satisfies the differential equation.

Thus, each of the functions is a solution of the given differential equation.

2). Verify that they are linearly independent.

$$W(f_1, f_2, f_3, f_4) = \begin{vmatrix} 1 & x & \cos x & \sin x \\ 0 & 1 & -\sin x & \cos x \\ 0 & 0 & -\cos x & -\sin x \\ 0 & 0 & \sin x & -\cos x \end{vmatrix} = \begin{vmatrix} 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \\ 0 & \sin x & -\cos x \end{vmatrix} =$$

$$\begin{vmatrix} -\cos x & -\sin x \\ \sin x & -\cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

Since $W(f_1, f_2, f_3, f_4) \neq 0$ for any x in $(-\infty, \infty)$, then functions f_1, f_2, f_3, f_4 are linearly independent.

Thus, since the functions f_1, f_2, f_3, f_4 satisfy the given differential equation and are linearly independent, they form the fundamental set of solutions, and we can write the general solution of the given equation as

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x) + C_4 y_4(x) = C_1 + C_2 x + C_3 \cos x + C_4 \sin x.$$

Problem 2. Solve the following initial-value problem

$$y'' + 4y = 17e^x \sin(2x), \quad y(0) = 1, \quad y'(0) = 0.$$

Solution. This is a nonhomogeneous equation. Thus, its solution is

$$y(x) = y_c(x) + y_p(x),$$

where $y_c(x)$ is a general solution of the corresponding homogeneous equation

$$y_c'' + 4y_c = 0$$

and $y_p(x)$ is a particular solution of the nonhomogeneous equation

$$y_p'' + 4y_p = 17e^x \sin(2x).$$

First let us find a general solution of the homogeneous equation. The auxiliary equation for the corresponding homogeneous differential equation is the following

$$m^2 + 4 = 0.$$

Thus, the roots of the auxiliary equation are $m_1 = 2i$, $m_2 = -2i$ and the general solution of the corresponding homogeneous problem is

$$y_c(x) = C_1 \cos(2x) + C_2 \sin(2x).$$

where C_1 and C_2 are arbitrary constants.
Since the right-hand side part $17e^x \sin(2x)$ is the product of exponential and trigonometric functions, and $1 + 2i$ is not a root of the auxiliary equation, we find particular solution as

$$y_p(x) = e^x (A \cos(2x) + B \sin(2x)),$$

where A and B are unknown coefficients. To find these coefficients, we substitute $y_p(x)$ into the nonhomogeneous equations. Since

$$y_p'(x) = e^x ((A + 2B) \cos(2x) + (B - 2A) \sin(2x))$$

and

$$y_p''(x) = e^x ((4B - 3A) \cos(2x) - (4A + 3B) \sin(2x)),$$

from the given differential equation we receive the following identity

$$e^x ((4B-3A) \cos(2x) - (4A+3B) \sin(2x)) + 4e^x (A \cos(2x) + B \sin(2x)) \equiv 17e^x \sin(2x).$$

From this,

$$(4B + A) \cos(2x) - (4A - B) \sin(2x) \equiv 17 \sin(2x).$$

Thus, we have the system of two linear algebraic equations

$$4B + A = 0, \quad B - 4A = 17.$$

Hence

$$A = -4, \quad B = 1$$

and

$$y_p(x) = e^x (\sin(2x) - 4 \cos(2x)).$$

Hence, the general solution of the nonhomogeneous equation is

$$y(x) = C_1 \cos(2x) + C_2 \sin(2x) + e^x (\sin(2x) - 4 \cos(2x)).$$

Next we find unknown constants C_1 and C_2 from the initial conditions

$$y(0) = C_1 - 4 = 1, \quad \text{and} \quad C_1 = 5.$$

Since

$$y'(x) = 2C_2 \cos(2x) - 2C_1 \sin(2x) + e^x(9\sin(2x) - 2\cos(2x))$$

then

$$y'(0) = 2C_2 - 2 = 0 \quad \text{and} \quad C_2 = 1.$$

Thus, we get the unique solution of the given initial-value problem

$$y(x) = (5 - 4e^x)\cos(2x) + (1 + e^x)\sin(2x).$$

Problem 3. Solve the differential equation

$$y'' - 2y' + y = \frac{e^x}{x}.$$

Solution. This is a nonhomogeneous equation. Thus its solution

$$y(x) = y_c(x) + y_p(x),$$

where $y_c(x)$ is a general solution of the corresponding homogeneous equation

$$y_c'' - 2y_c' + y_c = 0$$

and $y_p(x)$ is any particular solution of the nonhomogeneous equation

$$y_p'' - 2y_p' + y_p = \frac{e^x}{x}.$$

First let us find a general solution of the homogeneous equation. The auxiliary equation for this differential equation is the following

$$m^2 - 2m + 1 = (m - 1)^2 = 0.$$

Thus, there is one root of multiplicity 2 of the auxiliary equation $m_1 = m_2 = 1$ and the general solution of the corresponding homogeneous equation is

$$y_c(x) = e^x(C_1 + C_2 x),$$

where C_1 and C_2 are arbitrary constants.

Since the right-hand side part $\frac{e^x}{x}$ is not a product or sum of polynomials, exponents, sin or cos, we apply the method of variation of parameters to find the general solution of the given equation in the form

$$y(x) = e^x(C_1(x) + xC_2(x)),$$

where unknown functions $C_1(x)$ and $C_2(x)$ are determined from the following system of equations

$$\begin{aligned} e^x C_1'(x) + xe^x C_2'(x) &= 0 \\ e^x C_1'(x) + (1+x)e^x C_2'(x) &= \tfrac{e^x}{x} \end{aligned} \quad \text{or} \quad \begin{aligned} C_1'(x) + xC_2'(x) &= 0 \\ C_1'(x) + (1+x)C_2'(x) &= \tfrac{1}{x}. \end{aligned}$$

From this system we find

$$C_2'(x) = \frac{1}{x} \quad \text{and} \quad C_1'(x) = -xC_2'(x) = -1.$$

Thus

$$C_1(x) = \int (-1)dx = -x + C_1 \quad \text{and} \quad C_2(x) = \int \frac{1}{x}dx = \ln|x| + C_2.$$

Hence, the general solution

$$y(x) = e^x(C_1(x) + xC_2(x)) = e^x(-x + C_1 + x(\ln|x| + C_2)) =$$
$$e^x(C_1 + (C_2 - 1)x + x\ln|x|).$$

Problem 4. Solve the following boundary-value problem

$$y''(e^x + 1) + y' = 0, \quad y(0) = 0, \quad y'(2) = 1.$$

Solution. This is an equation with the missing dependent variable $y(x)$. Thus, we can reduce it to a first order differential equation. Let us denote $y'(x) = u(x)$. Then $y''(x) = u'(x)$, and we get the following first order differential equation

$$(e^x + 1)u' + u = 0.$$

Dividing both part by $(e^x + 1)u$, we get

$$\frac{u'}{u} = -\frac{1}{e^x + 1}.$$

Integrating both parts of the last equation leads to

$$\ln|u| = -\int \frac{dx}{e^x + 1} = -\int \frac{e^{-x}dx}{1 + e^{-x}} = \ln(1 + e^{-x}) + c_1,$$

from which

$$|u| = e^{c_1}(1 + e^{-x}) \quad \text{and} \quad u = C_1(1 + e^{-x}).$$

Since $u = y'$ then

$$y(x) = C_1 \int (1 + e^{-x})dx = C_1(x - e^{-x}) + C_2.$$

From the boundary conditions,

$$y(0) = -C_1 + C_2 = 0, \quad y'(2) = C_1(1 + e^{-2}) = 1.$$

From this, $C_1 = \frac{1}{1+e^{-2}} = \frac{e^2}{e^2+1}$ and $C_2 = C_1 = \frac{e^2}{e^2+1}$. Thus, the solution of given boundary-value problem is

$$y(x) = \frac{e^2}{e^2 + 1}(1 + x - e^{-x}).$$

Problem 5. Given the equation of free undamped motion

$$\frac{d^2x}{dt^2} + 36x = 0, \quad x(0) = 4, \quad x'(0) = -18,$$

find the amplitude and the phase angle of free vibrations.

Solution. The general solution of the given equation is

$$x(t) = C_1 \cos 6t + C_2 \sin 6t.$$

From the initial conditions we find

$$x(0) = C_1 = 4.$$

Since $x'(t) = -6C_1 \sin 6t + 6C_2 \cos 6t$ then

$$x'(0) = 6C_2 = -18 \quad \text{and } C_2 = -3.$$

From the alternative form of the equation of free undamped motion

$$x(t) = 4\cos 6t - 3\sin 6t = \sqrt{4^2 + (-3)^2}\left(\frac{4}{\sqrt{4^2+(-3)^2}}\cos 6t - \frac{3}{\sqrt{4^2+(-3)^2}}\sin 6t\right) =$$

$$5(\sin\varphi\cos 6t + \cos\varphi\sin 6t) = 5\sin(6t+\varphi),$$

where $\sin\varphi = \frac{4}{5}$ and $\cos\varphi = -\frac{3}{5}$, we find amplitude

$$A = 5$$

and phase angle

$$\varphi = \arctan\left(-\frac{4}{3}\right), \quad \frac{\pi}{2} < \varphi < \pi.$$

Bonus question. Find the general solution of the following nonlinear differential equation

$$y^3 y'' = 1.$$

Solution. This is an equation with the missing independent variable x. Thus, we can reduce it to a first order differential equation. Let us denote $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy}\cdot\frac{dy}{dx} = u\frac{du}{dy}$, and by substitution $u\frac{du}{dy}$ into the given differential equation instead of y'', we get the following first order differential equation

$$y^3\left(u\frac{du}{dy}\right) = 1 \quad \text{or} \quad udu = \frac{dy}{y^3}.$$

Integrating both parts of the last equation leads to

$$\frac{1}{2}u^2 = -\frac{1}{2y^2} + \frac{C_1}{2} \quad \text{or} \quad u^2 = \frac{C_1 y^2 - 1}{y^2},$$

from which

$$u = \pm\sqrt{\frac{C_1y^2 - 1}{y^2}} = \frac{\sqrt{C_1y^2 - 1}}{y}.$$

Since $u = \frac{dy}{dx}$ then

$$\frac{dy}{dx} = \frac{\sqrt{C_1y^2 - 1}}{y} \qquad \text{or} \qquad \frac{ydy}{\sqrt{C_1y^2 - 1}} = dx$$

Integrating the last equation leads to

$$\frac{\sqrt{C_1y^2 - 1}}{C_1} = x + c_2$$

or

$$C_1y^2 - 1 = (C_1x + C_2)^2,$$

where C_1 and C_2 are arbitrary constants.