MAT3379 (Winter 2013)

Assignment 3

Due date (Assignment 3): 8 April 2011

FINAL VERSION

The following questions will be marked: Q2, Q3b), c), Q4

Total number of points for Assignment 3: 12

Q1. (Theoretical Question).
Maximum Likelihood Estimation for AR(p) models.

Consider AR(1) model $X_t = \phi X_{t-1} + Z_t$, where Z_t are i.i.d. normal random variables with mean zero and variance σ^2. Derive MLE for ϕ and σ^2. (Hint: You should get formulas as in Example 6.6, but I need to see calculations).

Solution to Q1:

The likelihood function is given by (see Section 6.3)

$$L(\sigma, \phi) = \frac{1}{(\sqrt{2\pi})^n \sigma^n} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \phi X_{i-1})^2 \right). \tag{1}$$

Hence, the log-likelihood is

$$\ell(\sigma, \phi) = \log L(\sigma, \phi) = -n \log \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \phi X_{i-1})^2.$$

Taking derivative w.r.t. ϕ yields

$$\sum_{i=1}^{n} X_{i-1}(X_i - \phi X_{i-1}) = 0$$

and

$$\hat{\phi}_{\text{MLE}} = \frac{\sum_{i=1}^{n} X_{i-1} X_i}{\sum_{i=1}^{n} X_{i-1}^2}.$$

Taking derivative w.r.t. σ yields

$$-\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (X_i - \phi X_{i-1})^2 = 0$$

and

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \phi X_{i-1})^2.$$

Replacing ϕ with the MLE yields

$$\hat{\sigma}^2_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \hat{\phi}_{\text{MLE}} X_{i-1})^2.$$

Marking scheme for Q1:

This question will not be marked.

Q2. (Practical/Theoretical Question - 2 points)

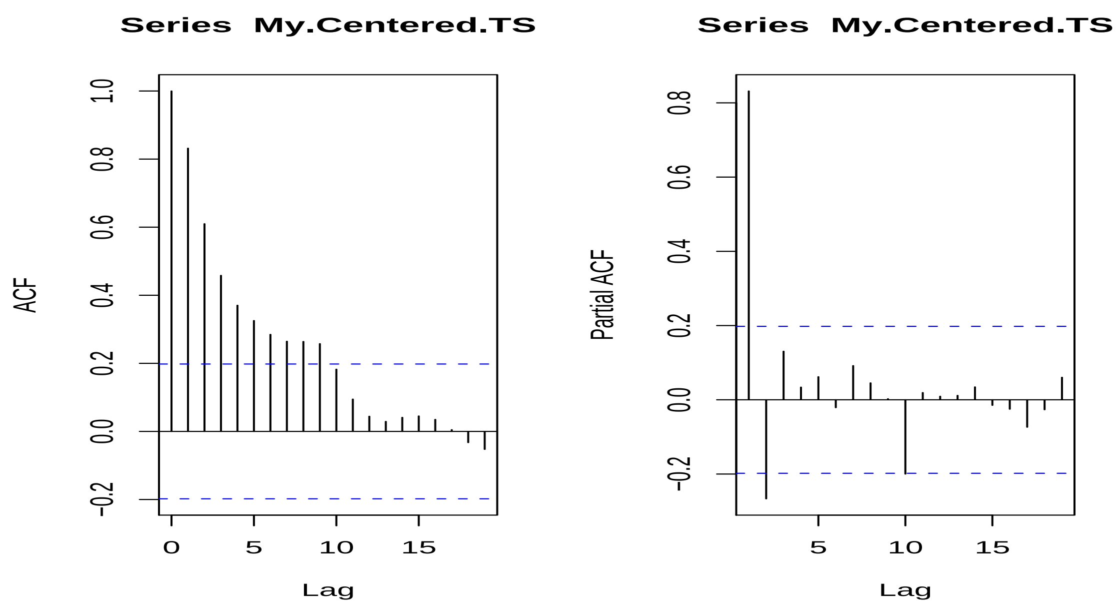
(a) Type
`My.TS<-LakeHuron; help(LakeHuron); mean=mean(My.TS);`
`My.Centered.TS<-My.TS-mean(My.TS);`
The first command loads data set `LakeHuron` which is in-built in `R`. The second command shows description of the data set. The third command centers your data set.

(b) Fit AR(2) model using the Yule-Walker estimator. Obtain $\hat{\phi}_1$, $\hat{\phi}_2$, $\hat{\sigma}^2$.
`fit.ar<-ar(My.Centered.TS,method="yule-walker");`
We did this in class!

(c) Verify that the command `ar` leads to the correct Yule-Walker estimator.
  - Type
    `ACF<-acf(LakeHuron)`
    and read $\hat{\rho}_X(1)$ and $\hat{\rho}_X(2)$. Type `var(LakeHuron)` to get $\hat{\gamma}_X(0)$. Using these information, compute $\hat{\gamma}_X(1)$, $\hat{\gamma}_X(2)$.
  - Create a vector $(\hat{\gamma}_X(1), \hat{\gamma}_X(2))$ and call it `gamma.vector`.
  - Create a matrix $\hat{\Gamma}_2$ and call it `Gamma.matrix`.
  - Type
    `solve(Gamma.matrix)%*%gamma.vector;`
    and compare the obtained values with part (b).

Solution to Q2:

By typing
`acf(My.Centered.TS); pacf(My.Centered.TS)`
we obtain



PACF suggest that AR(2) is a good fit, with $\phi_1 > 0$ and $\phi_2 < 0$.

By typing

```
fit.ar<-ar(My.Centered.TS,method="yule-walker");
fit.ar
```

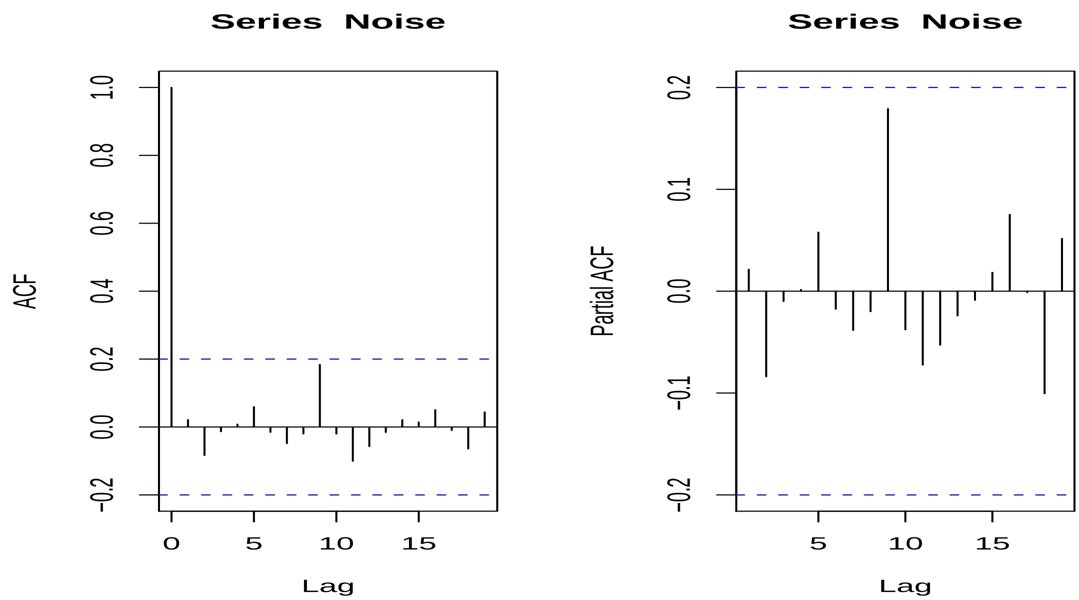
we obtain

```
Call:
ar(x = My.Centered.TS, method = "yule-walker")

Coefficients:
      1        2
 1.0538  -0.2668

Order selected 2  sigma^2 estimated as  0.5075
```

Thus,

$$\hat{\phi}_1 = 1.0538\ , \quad \hat{\phi}_2 = -0.2668\ , \quad \hat{\sigma}^2 = 0.5075\ . \tag{2}$$

The graphs show ACF and PACF of residuals.



They suggest that the residuals form a white noise - AR(2) fir is appropriate.

By typing

```
ACF<-acf(LakeHuron); ACF; var(LakeHuron);
```

we obtain

```
Autocorrelations of series LakeHuron, by lag

     0      1      2      3      4      5      6      7      8      9     10
 1.000  0.832  0.610  0.458  0.371  0.326  0.285  0.265  0.264  0.258  0.183
    11     12     13     14     15     16     17     18     19
 0.095  0.044  0.029  0.041  0.045  0.035  0.005 -0.033 -0.053
```

and

```
1.737911
```

We can read that

```
rho1=0.832; rho2=0.610;
```

We fill in the matrix and the vector

```
gamma0=var(LakeHuron); gamma1=rho1*gamma0; gamma2=rho2*gamma0;
Gamma.matrix=matrix(c(gamma0,gamma1,gamma1,gamma0),byrow=T,2,2);
gamma.vector=c(gamma1,gamma2);
solve(Gamma.matrix)%*%gamma.vector;
```

The result is

```
          [,1]
[1,]  1.0542732 [2,] -0.2671553  (***)
```

which agrees with (2) (there is some difference due to rounding).

Marking scheme for Q2:

2 points for getting the same values in (2) and (***)

Q3. (Theoretical Question - 5 points). GARCH models.

Consider a GARCH model

$$X_t = \sigma_t Z_t, \qquad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2,$$

where Z_t are i.i.d. random variables with mean 0 and variance 1.

(a) Compute $\mathrm{E}[X_t^2]$.

(b) Compute $\mathrm{E}[X_{t-1}X_t^2]$. What happens if we assume additionally that Z_t are standard normal?

(c) Show that

$$\mathrm{E}[\sigma_t^4] = \frac{\alpha_0^1}{1-\alpha_1}\frac{1+\alpha_1}{1-3\alpha_1^2}. \tag{3}$$

Solution to Q3:

(a) Compute $\mathrm{E}[X_t^2]$.
Recall that for fixed t, σ_t and Z_t are independent.
Method 1:

$$\mathrm{E}[X_t^2] = \mathrm{E}[\sigma_t^2 Z_t^2] = \mathrm{E}[\sigma_t^2]\mathrm{E}[Z_t^2] = \mathrm{E}[\sigma_t^2] = \mathrm{E}[\alpha_0 + \alpha_1 X_{t-1}^2] = \alpha_0 + \alpha_1 \mathrm{E}[X_{t-1}^2].$$

Since the sequence is stationary, $\mathrm{E}[X_{t-1}^2] = \mathrm{E}[X_t^2]$. Hence, we have to solve

$$\mathrm{E}[X_t^2] = \alpha_0 + \alpha_1 \mathrm{E}[X_t^2]$$

which leads to

$$\mathrm{E}[X_t^2] = \frac{\alpha_0}{1-\alpha_1}.$$

Method 2: Use the stationary representation

$$\mathrm{E}[X_t^2] = \alpha_0 \sum_{j=0}^{\infty} \alpha_j \mathrm{E}\left[Z_t^2 Z_{t-1}^2 \cdots Z_{t-j}^2\right] = \alpha_0 \sum_{j=0}^{\infty} \alpha_j = \frac{\alpha_0}{1-\alpha_1}$$

since Z_t are independent centered with variance one.

(b) Compute $\mathrm{E}[X_{t-1}X_t^2]$. What happens if we assume additionally that Z_t are standard normal?
The idea is to use definitions $X_t = \sigma_t Z_t$ and $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$.

$$\mathrm{E}[X_{t-1}X_t^2] = \mathrm{E}[\sigma_{t-1}Z_{t-1}\sigma_t^2 Z_t^2] = \mathrm{E}[Z_t^2]\mathrm{E}[\sigma_{t-1}Z_{t-1}\sigma_t^2] = \mathrm{E}[\sigma_{t-1}Z_{t-1}\sigma_t^2]$$

since Z_t is independent of σ_t, σ_{t-1} and Z_{t-1} (volatility σ_t depends on the past noise $Z_{t-1}, Z_{t-2}, \ldots$ only).
Now, plug-in the definition of σ_t^2:

$$\mathrm{E}[X_{t-1}X_t^2] = \mathrm{E}[\sigma_{t-1}Z_{t-1}\sigma_t^2] = \mathrm{E}[\sigma_{t-1}Z_{t-1}(\alpha_0 + \alpha_1 X_{t-1}^2)] = \alpha_0\mathrm{E}[\sigma_{t-1}Z_{t-1}] + \alpha_1\mathrm{E}[\sigma_{t-1}Z_{t-1}X_{t-1}^2].$$

The first term vanishes since

$$\mathrm{E}[\sigma_{t-1}Z_{t-1}] = \mathrm{E}[\sigma_{t-1}]\mathrm{E}[Z_{t-1}] = 0.$$

The second term is

$$\alpha_1\mathrm{E}[\sigma_{t-1}Z_{t-1}X_{t-1}^2] = \alpha_1\mathrm{E}[\sigma_{t-1}Z_{t-1}\sigma_{t-1}^2 Z_{t-1}^2] = \alpha_1\mathrm{E}[\sigma_{t-1}^3 Z_{t-1}^3] = \alpha_1\mathrm{E}[\sigma_{t-1}^3]\mathrm{E}[Z_{t-1}^3].$$

If Z_t are standard normal then $\mathrm{E}[Z_{t-1}^3]$. Hence, under this assumption, $\mathrm{E}[X_{t-1}X_t^2] = 0$.

(c) Show that

$$\mathrm{E}[\sigma_t^4] = \frac{\alpha_0^2}{1-\alpha_1}\frac{1+\alpha_1}{1-3\alpha_1^2}.$$

Again, the idea is to use definitions $X_t = \sigma_t Z_t$ and $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$.

$$\mathrm{E}[\sigma_t^4] = \mathrm{E}[(\alpha_0 + \alpha_1 X_{t-1}^2)^2] = \alpha_0^2 + 2\alpha_0\alpha_1\mathrm{E}[X_{t-1}^2] + \alpha_1^2\mathrm{E}[X_{t-1}^4].$$

In (a) we evaluated $\mathrm{E}[X_{t-1}^2] = \mathrm{E}[X_t^2]$. We compute $\mathrm{E}[X_{t-1}^4]$:

$$\mathrm{E}[X_{t-1}^4] = \mathrm{E}[\sigma_{t-1}^4 Z_{t-1}^4] = \mathrm{E}[\sigma_{t-1}^4]\mathrm{E}[Z_{t-1}^4] = 3\mathrm{E}[\sigma_{t-1}^4] = 3\mathrm{E}[\sigma_t^4],$$

where we used independence between Z_{t-1} and σ_{t-1}, stationarity to conclude $\mathrm{E}[\sigma_t^4] = \mathrm{E}[\sigma_{t-1}^4]$ and assumption that Z_t are standard normal so that $\mathrm{E}[Z_{t-1}^4] = 3$.
Hence,

$$\mathrm{E}[\sigma_t^4] = \alpha_0^2 + 2\alpha_0\alpha_1\mathrm{E}[X_{t-1}^2] + \alpha^2\mathrm{E}[X_{t-1}^4] = \alpha_0^2 + 2\alpha_0\alpha_1\frac{\alpha}{1-\alpha_1} + 3\alpha_1^2\mathrm{E}[\sigma_t^4].$$

Solving for $\mathrm{E}[\sigma_t^4]$ we obtain (3).

Marking scheme for Q3:

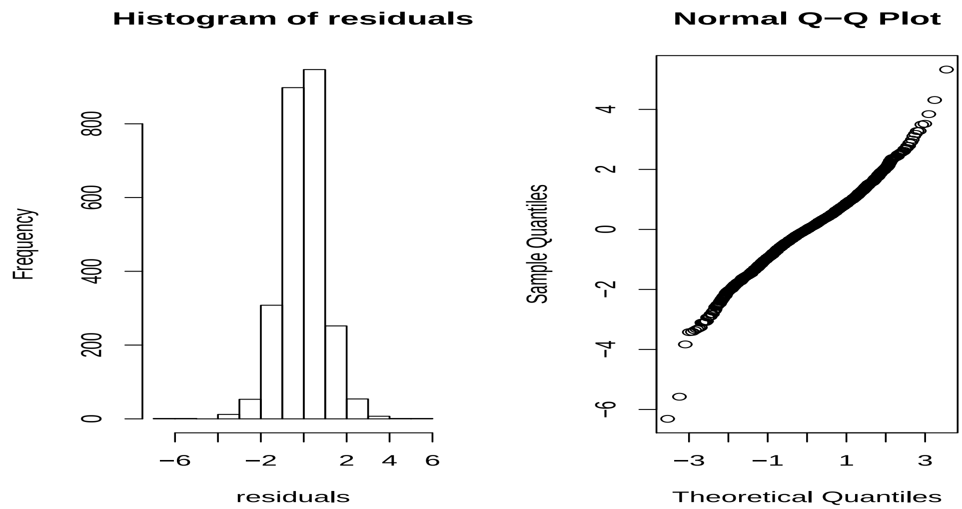


FIGURE 1. GARCH(0,1)=ARCH(1)

Part (a) will not be marked. Part (b): 1 point for some steps towards $\mathrm{E}[X_{t-1}X_t^2]$, 1 point for the expression

$$\mathrm{E}[X_{t-1}X_t^2] = \alpha_1 \mathrm{E}[\sigma_{t-1}^3]\mathrm{E}[Z_{t-1}^3],$$

1 point for the conclusion in case of normality. Total 3 points for part (b).

For part (c): 1 point for some steps towards the final answer. 1 point for the final answer. Total 2 points for part (c).

Q4. (Practical Question - 4 points). Fitting GARCH model. Estimation.

- Download data set `swiss`.
- Compute log-returns.

(a) Use function `garch` to fit GARCH model. Consider ARCH(1), ARCH(2) and GARCH(1,1) and choose the most suitable one. You have to analyse residuals. Check if the resulting model is stationary.
(b) Plot the estimated volatility.
(c) Predict the next observation for the volatility.

Solution to Q4:

(a) Log-returns:

```
library(tseries); Data=diff(log(swiss));
```

ARCH(1) fit.

```
fit.ARCH1<-garch(Data,order=c(0,1)); residuals<-fit.ARCH1$residuals;
par(mfrow=c(1,2)); hist(residuals); qqnorm(residuals);
```

Residuals are shown on Figure 1. They do not seem to be normal.

ARCH(2) fit.

```
fit.ARCH2<-garch(Data,order=c(0,2)); residuals<-fit.ARCH2$residuals;
par(mfrow=c(1,2)); hist(residuals); qqnorm(residuals);
```

Residuals are shown on Figure 2. They do not seem to be normal.

GARCH(1,1) fit.

```
fit.GARCH<-garch(Data,order=c(1,1)); residuals<-fit.GARCH$residuals;
par(mfrow=c(1,2)); hist(residuals); qqnorm(residuals);
```

Residuals are shown on Figure 3. They seem to be normal.

Choose GARCH(1,1) model.

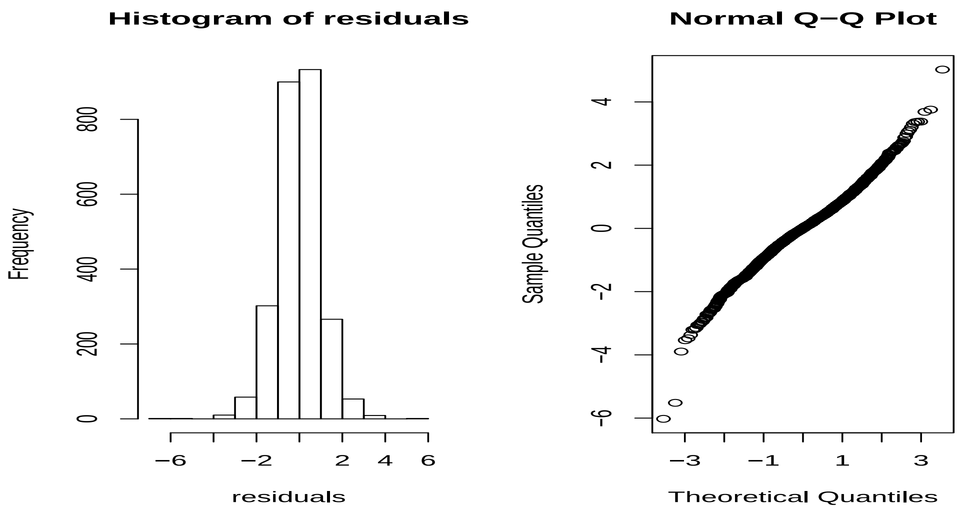


FIGURE 2. GARCH(0,2)=ARCH(2)

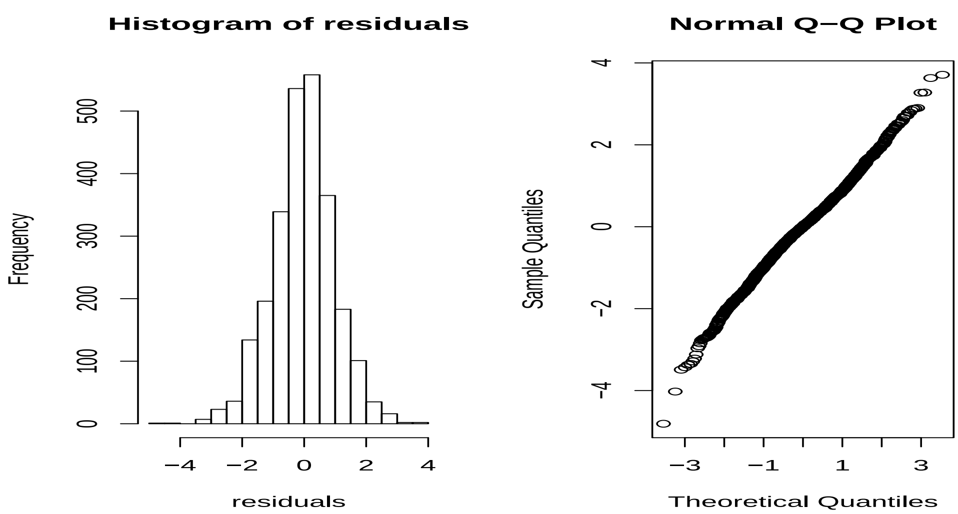


FIGURE 3. GARCH(1,1)

```
Coefficients<-fit.GARCH$coef; Coefficients;
          a0             a1             b1
1.929693e-07 2.802573e-02 9.680856e-01
```

Note that the sum of α_1 and β_1 is smaller than 1, so that the model is stationary.

(b)
```
volatility<-fit.GARCH$fitted.values; volatility<-volatility[,1];
par(mfrow=c(1,1)); plot.ts(volatility);
```

The predicted volatility is

$$\hat{\sigma}^2_{n+1} = \hat{\alpha}_0 + \hat{\alpha}_1 X_n^2 + \hat{\beta}_1 \hat{\sigma}^2_n.$$

Above, X_n is the last observation and σ_n is last estimated volatility.

(c)
```
n=length(Data); Xn=Data[n]; sigman=volatility[n];
```

The prediciton is:

```
Coefficients[1]+Coefficients[2]*Xn^2+Coefficients[3]*sigman^2
```

The answer is: 3.678404e-05.

Marking scheme for Q4:

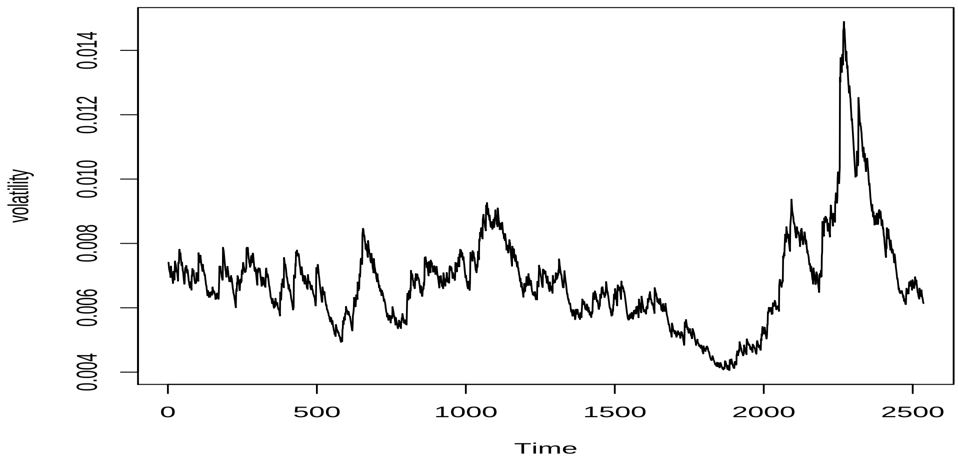


FIGURE 4. Volatility

Part (a): 1 point for correct choice of GARCH(1,1) based on residuals, 1 point for correct coefficients. Total 2 points for part (a). 1 point for part (b). 1 point for some steps in part (c), 1 point for the correct answer in part (c). Total: 5 points.

Q5. (Practical/Theoretical Question) The following exercise shows that it is hard to identify AR model with $p \geq 2$.

- Download `BadData.txt`. Denote by X your data set.

(a) Based on ACF and PACF argue that an AR(3) model can be chosen.

(b) Type

```
fit.ar<-ar(X,method="mle");
fit.ar;
```

What order has been selected? Denote this order by p.

(c) Use p from (b) and type

```
fit.arima<-arima(X,order=c(3,0,0));
fit.arima;
fit.arima1<-arima(X,order=c(p,0,0))
fit.arima1;
```

Why did MLE select p, not 3?

Solution to Q5:

(a) ACF and PACF are messy. Ignoring significant values for large lags of PACF, we can argue that an AR(3) model can be chosen.

(b) Type

```
fit.ar<-ar(X,method="mle"); fit.ar;
```

The output is

```
Call: ar(x = X, method = "mle")

Coefficients:
      1        2        3        4        5
 0.5353  -0.4480   0.2233  -0.0800  -0.2022

Order selected 5  sigma^2 estimated as  0.7512
```

The order selected is 5.

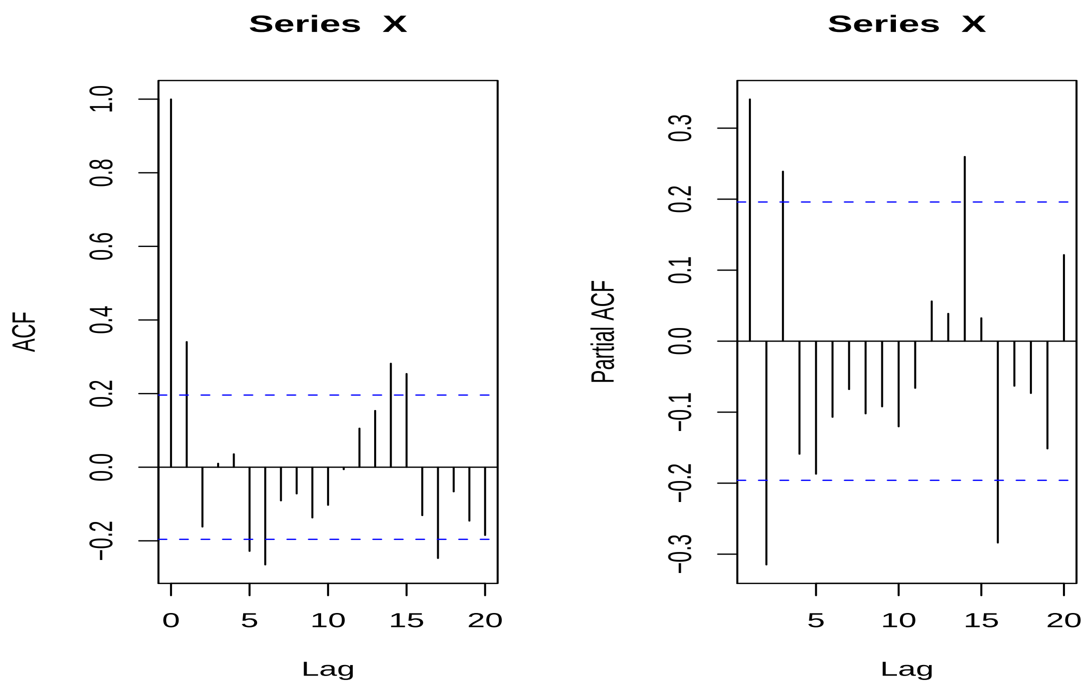


FIGURE 5. ACF and PACF

`p=5;`

(c) Use p from (b) and type

```
fit.arima<-arima(X,order=c(3,0,0)); fit.arima;
fit.arima1<-arima(X,order=c(p,0,0)) fit.arima1;
```

MLE selects $p = 5$ since the AIC is smaller. What happens here is that the covariances and partial covariances are messy, we ignored them in (a), but they influence MLE.

Marking scheme for Q5:

This question will not be marked.