MAT3379 (Winter 2013)

Assignment 1

Due date (Assignment 1): Monday, 4 January 2013

FINAL VERSION

Q1. (4 points) Let $\{Z_t\}$ be an IID sequence of normal random variables with mean 0 and variance σ^2. Let a, b, c be constants. Which of the following processes are stationary? Evaluate mean and autocovariance function.

(a) $X_t = Z_t \cos(at) + Z_{t-1} \sin(bt)$.
(b) $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$.
(c) $X_t = Z_t Z_{t-1}$.

Solution to Q1:

Note that $\mathrm{E}[Z_t Z_{t+m}] = \mathrm{E}[Z_t]\mathrm{E}[Z_{t+m}] = 0$ for all $m \neq 0$ and

(a) We have

$$\mathrm{E}[X_t] = \cos(at)\mathrm{E}[Z_t] + \sin(bt)\mathrm{E}[Z_{t-1}] = 0$$

since $\mathrm{E}[Z_t] = 0$. Hence, the mean does not depend on t.
Also,

$$\begin{aligned}\gamma(t, t+h) = \mathrm{E}[X_t X_{t+h}] =& \cos(at)\cos(a(t+h))\mathrm{E}[Z_t Z_{t+h}] + \sin(at)\sin(a(t+h))\mathrm{E}[Z_{t-1} Z_{t+h-1}] + \\ &+ \cos(at)\sin(a(t+h))\mathrm{E}[Z_t Z_{t+h-1}] + \sin(at)\cos(a(t+h))\mathrm{E}[Z_{t-1} Z_{t+h}].\end{aligned}$$

For $h = 0$ we get

$$\begin{aligned}\gamma(t,t) = \mathrm{E}[X_t^2] =& \cos^2(at)\mathrm{E}[Z_t^2] + \sin^2(at)\mathrm{E}[Z_{t-1}^2] + \\ &+ \cos(at)\sin(at)\mathrm{E}[Z_t Z_{t-1}] + \sin(at)\cos(at)\mathrm{E}[Z_{t-1} Z_t] \\ =& \cos^2(at)\sigma^2 + \sin^2(at)\sigma^2 = \sigma^2.\end{aligned}$$

Hence, the variance does not depend on t.
If we take $h = 1$ we obtain

$$\gamma(t, t+1) = \sigma^2 \cos(at)\sin(a(t+1)).$$

This expression cannot be simplified further. Covariance at lag $h = 1$ depends on t. The sequence is not stationary.

Marking: This part will not be marked.

(b) We have

$$\mathrm{E}[X_t] = \cos(ct)\mathrm{E}[Z_1] + \sin(ct)\mathrm{E}[Z_2] = 0$$

since $\mathrm{E}[Z_t] = 0$. Hence, the mean does not depend on t.
Also,

$$\begin{aligned}\gamma(t, t+h) = \mathrm{E}[X_t X_{t+h}] =& \cos(ct)\cos(c(t+h))\mathrm{E}[Z_1^2] + \sin(ct)\sin(c(t+h))\mathrm{E}[Z_2^2] + \\ &+ \cos(ct)\sin(c(t+h))\mathrm{E}[Z_1 Z_2] + \sin(ct)\cos(c(t+h))\mathrm{E}[Z_1 Z_2] \\ =& \cos(ct)\cos(c(t+h))\mathrm{E}[Z_1^2] + \sin(ct)\sin(c(t+h))\mathrm{E}[Z_2^2] \\ =& \sigma^2 \{\cos(ct)\cos(c(t+h)) + \sin(ct)\sin(c(t+h))\} = \sigma^2 \cos(ch).\end{aligned}$$

Hence, the covariance function does not depend on t. We are allowed to write $\gamma(t, t+h) = \gamma(h) = \sigma^2 \cos(ch)$. The sequence is stationary.

Marking: 0.5 point for correct computation of the mean; 1 point for correct computation of covariance (you must provide general formula for arbitrary h); 0.5 point for correct conclusion about stationarity. Total: 2 points.

(c) We have

$$\mathrm{E}[X_t] = \mathrm{E}[Z_t Z_{t-1}] = 0$$

since $\mathrm{E}[Z_t] = 0$. Hence, the mean does not depend on t. Also

$$\gamma(t, t+h) = \mathrm{E}[X_t X_{t+h}] = \mathrm{E}[Z_t Z_{t-1} Z_{t+h} Z_{t+h-1}].$$

If $h = 0$, then $\gamma(t,t) = \mathrm{E}[Z_t^2]\mathrm{E}[Z_{t-1}^2] = \sigma^2\sigma^2 = \sigma^4$. If $h \geq 1$, then none of the indices $t, t-1, t+h, t+h-1$ agree. Hence, $\gamma(t, t+h) = 0$. Hence, the covariance function does not depend on t. Random variables X_t are uncorrelated (but dependent !). The sequence is stationary.

Marking: 0.5 point for correct computation of the mean; 1 point for correct computation of covariance (you must provide general formula for arbitrary h); 0.5 point for correct conclusion about stationarity. Total: 2 points.

Q2. (0 points) Let $\{Z_t\}$ be an IID sequence of normal random variables with mean 0 and variance $\sigma^2 = 1$. Define

$$X_t = \begin{cases} Z_t, & t \quad \text{even}, \\ (Z_{t-1}^2 - 1)/\sqrt{2}, & t \quad \text{odd}. \end{cases}$$

Show that $\{X_t\}$ is $\mathrm{WN}(0,1)$, but it is not $\mathrm{IID}(0,1)$.

Solution to Q2:

We have

$$\mathrm{E}[X_t] = \begin{cases} \mathrm{E}[Z_t] = 0, & t \quad \text{even}, \\ \mathrm{E}\left[(Z_{t-1}^2 - 1)/\sqrt{2}\right] = 0, & t \quad \text{odd}. \end{cases}$$

The latter equality follows from

$$\mathrm{E}\left[(Z_{t-1}^2 - 1)/\sqrt{2}\right] = \frac{1}{\sqrt{2}}\mathrm{E}\left[(Z_{t-1}^2 - 1)\right] = 0$$

since $\mathrm{E}[Z_{t-1}^2] = \mathrm{Var}(Z_{t-1}) = 1$. Hence, $\mathrm{E}[X_t] = 0$ for all indices t.

Now, take t even. Then for $k \geq 1$ (note that the index $t + 2k$ is even as well, whereas the index $t + 2k - 1$ is odd)

$$\gamma(t, t+2k) = \mathrm{E}[Z_t Z_{t+2k}] = 0$$

and

$$\gamma(t, t+2k-1) = \frac{1}{\sqrt{2}}\mathrm{E}[Z_t(Z_{t+2k-1}^2 - 1)] = \frac{1}{\sqrt{2}}\mathrm{E}[Z_t]\mathrm{E}[(Z_{t+2k-2}^2 - 1)] = 0.$$

However, this is only correct for $k \geq 2$. If $k = 1$, then

$$\gamma(t, t+1) = \frac{1}{\sqrt{2}}\mathrm{E}[Z_t(Z_t^2 - 1)] = \frac{1}{\sqrt{2}}\left\{\mathrm{E}[Z_t^3] - \mathrm{E}[Z_t]\right\} = 0.$$

Similar computation is valid for t odd. Hence, the covariance is 0 and the sequence is a white noise.

On the other hand, the sequence is not i.i.d. because:

- For t even, random variables $X_t = Z_t$ and $X_{t+1} = \frac{1}{\sqrt{2}}(Z_t^2 - 1)$ are dependent.
- Furthermore, Z_t have normal distribution, and Z_t^2 has χ^2 distribution.

Marking: This part will not be marked.

Q3. (0 points) Let $\{Z_t\}$ be an IID sequence with mean 0 and variance σ^2. Let $\{Y_t\}$ be a stationary sequence with a covariance function $\gamma_Y(k)$. Assume also that the sequences $\{Z_t\}$ and $\{Y_t\}$ are independent from each other. Define $X_t = Y_t Z_t$.

Verify that for $k \geq 1$ we have $\mathrm{Cov}(X_t, X_{t+k}) = 0$ and $\mathrm{Cov}(X_t^2, X_{t+k}^2) \neq 0$ (at least not for all t and k), that is $\{X_t\}$ is a white noise but not IID.

Solution to Q3:

Note that $\mathrm{E}[X_t] = \mathrm{E}[Y_t]\mathrm{E}[Z_t] = 0$. We have for $k \neq 0$,

$$\gamma_X(t, t+k) = \mathrm{Cov}(X_t, X_{t+k}) = \mathrm{E}[X_t X_{t+k}] = \mathrm{E}[Z_t]\mathrm{E}[Z_{t+k}]\mathrm{E}[Y_t Y_{t+k}] = 0.$$

On the other hand for $k \neq 0$,

$$\begin{aligned}\gamma_{X^2}(t, t+k) =& \mathrm{Cov}(X_t^2, X_{t+k}^2) = \mathrm{E}[X_t^2 X_{t+k}^2] - \mathrm{E}[X_t^2]\mathrm{E}[X_{t+k}^2] \\ =& \mathrm{E}[Y_t^2 Y_{t+k}^2]\mathrm{E}[Z_t^2]\mathrm{E}[Z_{t+k}^2] - \mathrm{E}[Y_t^2]\mathrm{E}[Y_{t+k}^2]\mathrm{E}[Z_t^2]\mathrm{E}[Z_{t+k}^2] \\ =& \sigma^4 \left\{\mathrm{E}[Y_t^2 Y_{t+k}^2] - \mathrm{E}[Y_t^2]\mathrm{E}[Y_{t+k}^2]\right\} = \sigma^4 \mathrm{Cov}(Y_t^2, Y_{t+k}^2).\end{aligned}$$

Hence, whenever Y_t is a dependent sequence, then $\mathrm{Cov}(Y_t^2, Y_{t+k}^2) \neq 0$ at least for some $k \neq 0$, so that $\gamma_{X^2}(t, t+k) \neq 0$.

Marking: This part will not be marked.

Q4. (8 points) For the following processes, compute autocovariance function.

(a) AR(2) - use the recursive method;
(b) ARMA$(1,1)$ - you may use the linear representation from lectures;
(c) ARMA$(1,2)$ - derive a linear representation first!

Solution to Q4:

(a) (You may refer to Example 3.12 in Lecture Notes).
For AR(p) models $p \geq 2$, one applies a recursive method.
Take AR(2) equation $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$. Multiply both sides by X_{t-h} and apply the expected value to get

$$\mathrm{E}[X_t X_{t-h}] = \phi_1 \mathrm{E}[X_{t-1}X_{t-h}] + \phi_1 \mathrm{E}[X_{t-2}X_{t-h}] + \mathrm{E}[Z_t X_{t-h}].$$

Since $\mathrm{E}[X_t] = 0$, then $\mathrm{E}[X_t X_{t-h}] = \gamma_X(h)$, $\mathrm{E}[X_{t-1}X_{t-h}] = \gamma_{h-1}$ and $\mathrm{E}[X_{t-2}X_{t-h}] = \gamma_{h-2}$.
Also, for all $h \geq 1$, Z_t is independent of X_{t-h}. Hence, for $h \geq 1$

$$\mathrm{E}[Z_t X_{t-h}] = \mathrm{E}[Z_t]\mathrm{E}[X_{t-h}] = 0.$$

Hence, we obtain

$$\gamma_X(h) = \phi_1 \gamma_X(h-1) + \phi_2 \gamma_X(h-2).$$

(You cannot really proceed with induction as in AR(1) case).
We need to start the recursion by computing $\gamma_X(0) = \mathrm{Var}(X_t) = \sigma_X^2$ and $\gamma_X(1)$. To get $\gamma_X(1)$ we use again AR(2) equation, multiply by X_{t-1} and apply expectation to get

$$\gamma_X(1) = \phi_1 \gamma_X(0) + \phi_2 \gamma_X(1),$$

so that

$$\gamma_X(1)\frac{1-\phi_2}{\phi_1} = \gamma_X(0). \tag{1}$$

Now, we need to get $\gamma_X(0)$. Note that the previous procedure, that is multiplying the AR(2) equation by X_t and taking expectations would not work since X_t and Z_t are not independent. Take the AR(2) equation and apply variance at both sides:

$$\mathrm{Var}(X_t) = \mathrm{Var}(\phi_1 X_{t-1} + \phi_2 X_{t-2}) + \mathrm{Var}(Z_t) \tag{2}$$

(again, (X_{t-1}, X_{t-2}), and Z_t are independent.) However, X_{t-1} and X_{t-2} are dependent. We will use the formula $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$. Apply it to (2) to get

$$\mathrm{Var}(X_t) = \phi_1^2 \mathrm{Var}(X_{t-1}) + \phi_2^2 \mathrm{Var}(X_{t-2}) + 2\phi_1\phi_2 \mathrm{Cov}(X_{t-1}, X_{t-2}) + \sigma^2.$$

Equivalently,

$$\gamma_X(0) = (\phi_1^2 + \phi_2^2)\gamma_X(0) + 2\phi_1\phi_2\gamma_X(1) + \sigma^2, \tag{3}$$

or

$$\frac{\gamma_X(0)(1-\phi_1^2-\phi_2^2) - \sigma^2}{2\phi_1\phi_2} = \gamma_X(1), \tag{4}$$

Plugging-in the above expression into (1) yields

(5) $$\frac{\gamma_X(0)(1-\phi_1^2-\phi_2^2)-\sigma^2}{2\phi_1\phi_2}\frac{1-\phi_2}{\phi_1}=\gamma_X(0).$$

Finally,

(6) $$\gamma_X(h)=\phi_1\gamma_X(h-1)+\phi_2\gamma_X(h-2), \qquad h\geq 2$$

and $\gamma_X(1)$, $\gamma_X(0)$ are solutions to (1) and (5).
Marking: 2 points for recursion in equation (6); 1 point for recognizing that we need to evaluate both $\gamma_X(0)$ and $\gamma_X(1)$ (0.5 points each); 1 point for providing formulas for $\gamma_X(0)$ and $\gamma_X(1)$, either explicitly or as system of equations (1) and (5). Total: 4 points.

(b) See Example 3.11 in Lecture Notes.

Marking: This part will not be marked.

(c) We derive a linear representation for ARMA(1,2), You may follow the steps of Example 3.6. ARMA(1,2) is given by

(7) $$\phi(B)X_t=\theta(B)Z_t,$$

where $\phi(z)=1-\phi z$, $\theta(z)=1+\theta_1 z+\theta_2 z^2$. Define

$$\chi(z)=\frac{1}{\phi(z)}=\sum_{j=0}^{\infty}\phi^j z^j.$$

Take equation (7) and multiply both sides by $\chi(B)$:

$$\begin{aligned}\chi(B)\phi(B)X_t&=\chi(B)\theta(B)Z_t,\\ X_t&=\chi(B)\theta(B)Z_t,\end{aligned}$$

since $\chi(z)\phi(z)=1$ for all z. That is,

$$X_t=\chi(B)\theta(B)Z_t=\sum_{j=0}^{\infty}\phi^jB^j(1+\theta_1B+\theta_2B^2)Z_t=\sum_{j=0}^{\infty}\phi^jZ_{t-j}+\theta_1\sum_{j=0}^{\infty}\phi^jZ_{t-j-1}+\theta_2\sum_{j=0}^{\infty}\phi^jZ_{t-j-2}.$$

Until now everything was almost the same as for AR(1) or ARMA(1,1). Now, we want X_t to have a form $\sum_{j=0}^{\infty}\psi_jZ_{t-j}$. That is

$$\sum_{j=0}^{\infty}\psi_jZ_{t-j}=\sum_{j=0}^{\infty}\phi^jZ_{t-j}+\theta_1\sum_{j=0}^{\infty}\phi^jZ_{t-j-1}+\theta_2\sum_{j=0}^{\infty}\phi^jZ_{t-j-2}.$$

Re-write it as

$$\psi_0Z_t+\psi_1Z_{t-1}+\sum_{j=2}^{\infty}\psi_jZ_{t-j}=\phi^0Z_t+(\phi^1+\theta_1)Z_{t-1}+\sum_{j=2}^{\infty}(\phi^j+\theta_1\phi^{j-1}+\theta_2\phi^{j-2})Z_{t-j}.$$

We can identify coefficients as

(8) $$\psi_0=1, \qquad \psi_1=\phi(1+\theta_1), \qquad \psi_j=\phi^{j-2}(\phi^2+\theta_1\phi+\theta_2), \quad j\geq 2.$$

The above formula gives a linear representation for ARMA(1,1). The formula is obtained under the condition that $|\phi|<1$.

Having established the linear representation, we the general formula for covariance of linear processes:

$$\gamma_X(h)=\sigma_Z^2\sum_{j=0}^{\infty}\psi_j\psi_{j+h}=\sigma_Z^2\left\{\psi_0\psi_h+\psi_1\psi_{h+1}+\sum_{j=2}^{\infty}\psi_j\psi_{j+h}\right\}$$

For $h \geq 2$ we have

$$\gamma_X(h) = \sigma_Z^2 \left\{ \phi^{h-2}(\phi^2 + \theta_1\phi + \theta_2) + (\phi + \theta_1)\phi^{h-1}(\phi^2 + \theta_1\phi + \theta_2) + (\phi^2 + \theta_1\phi + \theta_2)^2 \sum_{j=2}^{\infty} \phi^{j-2}\phi^{j+h-2} \right\}.$$

For $h = 1$ we have

$$\gamma_X(1) = \sigma_Z^2 \left\{ (\phi + \theta_1) + \phi(1 + \theta_1)(\phi^2 + \theta_1\phi + \theta_2) + (\phi^2 + \theta_1\phi + \theta_2)^2 \sum_{j=2}^{\infty} \phi^{j-2}\phi^{j-1} \right\}.$$

You have to make this distinction between $h = 1$ and $h \geq 2$, since values ψ_1 and ψ_j, $j \geq 2$ are of the different form. You can simplify the formulas further.

Marking: 2 points for some derivation of linear representation; 1 point for correct coefficients in (8); 1 point for some steps in computation of covariances. Total: 4 points.

Q5. (2 points) Let $\{Z_t\}$ be WN$(0, \sigma^2)$. Determine if the following processes are stationary and causal.
(a) $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$.
(b) $X_t + 1.6X_{t-1} = Z_t - 0.42Z_{t-1} + 0.04Z_{t-2}$.

Solution to Q5:

Refer to page 6 of Lecture Notes:
(a) The autoregressive polynomial is $\phi(z) = 1 - 0.2z + 0.48z^2$. The roots are $(0.2 - \sqrt{-1.88})/2 = 0.1 - 0.69i$ and $(0.2 + \sqrt{-1.88})/ = 0.1 + 0.69i$. Modulus of both roots is smaller than one, so that there is a stationary but not causal solution.

Marking: 1 point for looking only at real solutions, 1 point for correct answer in the complex domain. Total: 2 points.

(b) The autoregressive polynomial is $\phi(z) = 1 - 1.6z$. The root is $1/1.6$. Hence, the sequence is stationary but not causal.

Marking: this part will not be marked.

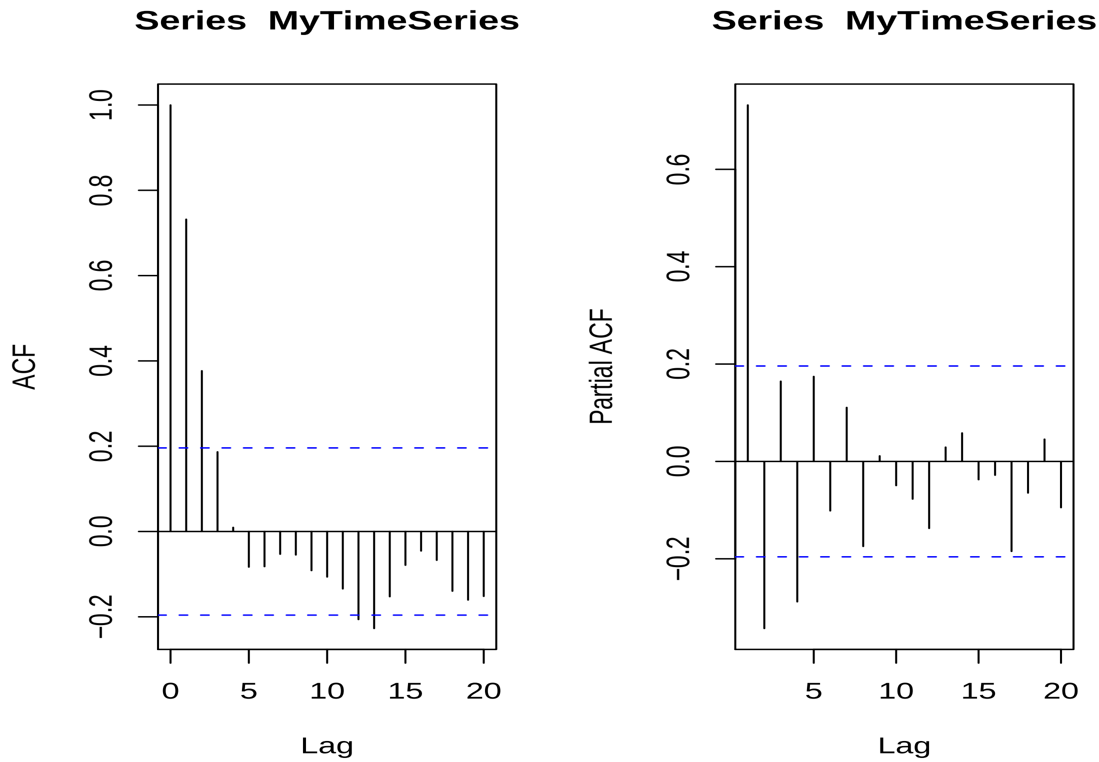
Q6. (Practical Question - 7 points) You may use codes available on the webpage.
(a) Generate ARMA(p, q) sequence X_t. You have to choose p, q as well as the required parameters. Make sure that the chosen parameters imply existence of a stationary solution.
(b) Identify the model using ACF and PACF. Include graphs of ACF and PACF (2 graphs).
(c) Add a linear or a polynomial trend m_t. The new sequence is $Y_t = m_t + X_t$.
(d) Estimate m_t using all three methods:
- parametric method;
- exponential smoothing;
- moving average smoothing with your chosen Q.

(e) For each of the three methods, plot Y_t and $\hat{m}_t$ on the same graphs (3 graphs).
(f) For each of the three methods, compute residuals $\tilde{X}_t = Y_t - \hat{m}_t$. Plot residuals (3 graphs).
(g) Analyse the residuals using ACF and PACF. Graph ACF and PACF for all three methods (6 graphs). Identify ARMA model. Compare with your identification in (b).

Solution to Q6:

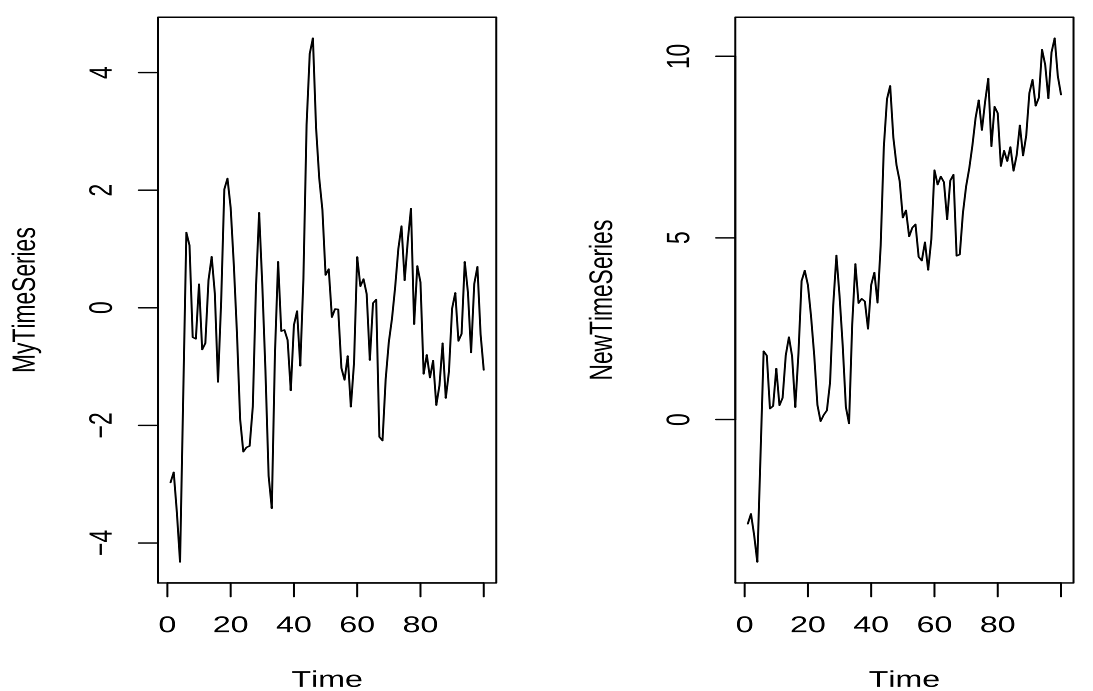
(a) I generated time series by typing:
`MyTimeSeries=ARMA(100,1,2,phi=0.3)`
(b) It is hard to identify the model from ACF and PACF. PACF suggests that there is MA part of order $q = 1$ (note that original time series was $q = 2$). ACF suggests that if there is no AR part, then MA part must

be of order $q = 4$. Consequently, it suggest that there is an autoregressive part in the model. I choose ARMA$(1,1)$.



(c) I added linear trend $m_t = 0.1t$:

```
NewTimeSeries=Lin.Trend(MyTimeSeries,0,0.1)
plot.ts(MyTimeSeries); plot.ts(NewTimeSeries)
```



(d), (e) I estimated linear parameters as

```
n=100
Time=c(1:n)
a.est=lm(NewTimeSeries~Time)$coefficients[1];
b.est=lm(NewTimeSeries~Time)$coefficients[2];
```

We obtain $\hat{a} = -0.48$, $b = 0.105$. Recall that the true parameters are $a = 0$, $b = 0.1$.

Then, I typed

```
Fitted.Lin.Trend=a.est+b.est*Time;
plot(Time,NewTimeSeries,type="l",col="black");
points(Time,Fitted.Lin.Trend,type="l",col="blue");
ResidualsLinear=NewTimeSeries-Fitted.Lin.Trend;
```

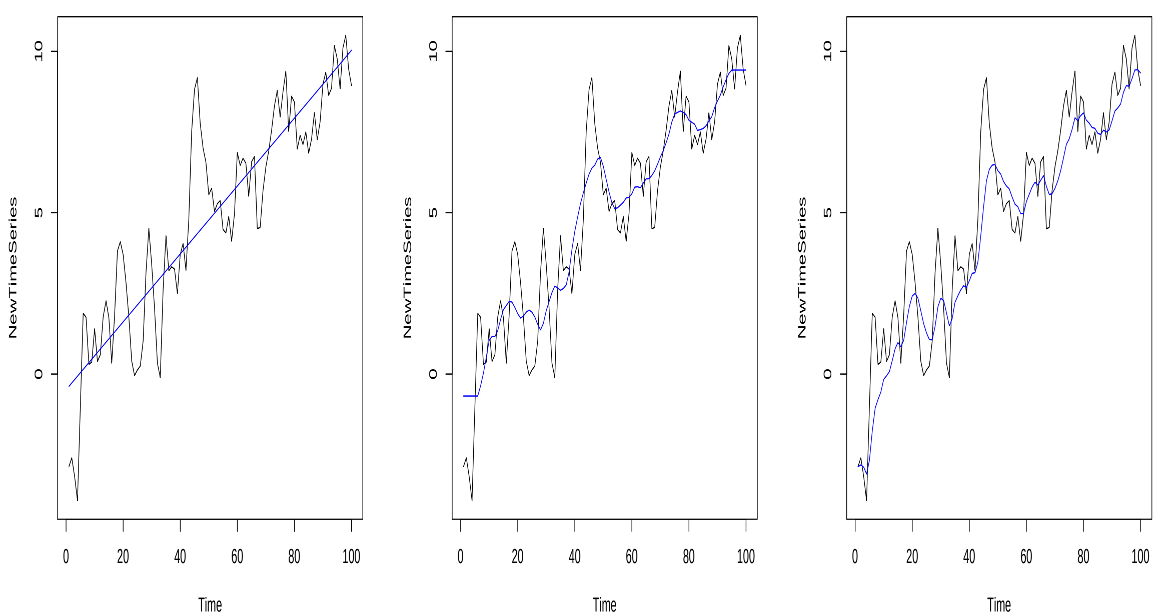
Then, I applied moving average smoothing as follows

```
MySmoothedTimeSeries=SmoothedTS(NewTimeSeries,5)
plot(Time,NewTimeSeries,type="l",col="black");
points(Time,MySmoothedTimeSeries,type="l",col="blue");
ResidualsMASmooth=NewTimeSeries-MySmoothedTimeSeries;
```

Then, I applied exponential smoothing as follows

```
AnotherSmoothedTimeSeries=ExpSmooth(NewTimeSeries,0.2)
plot(Time,NewTimeSeries,type="l",col="black");
points(Time,AnotherSmoothedTimeSeries,type="l",col="blue");
ResidualsExpSmooth=NewTimeSeries-AnotherSmoothedTimeSeries;
```

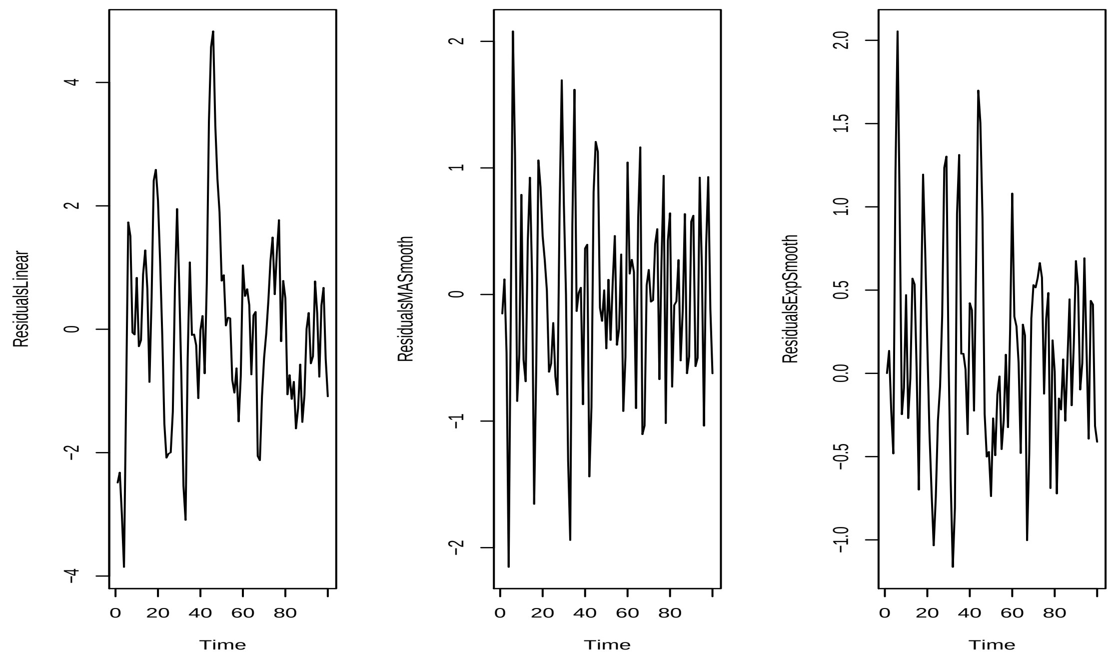
In both case, parameters $Q = 5$ in the MA smoothing and $\alpha = 0.2$ in the exponential smoothing are chosen by trying several parameters. Idea: you should choose parameters such that the smoothed time series does not follow too closely the original one.



(f) We plot residuals in three above cases:

```
plot.ts(ResidualsLinear);
plot.ts(ResidualsMASmooth);
```

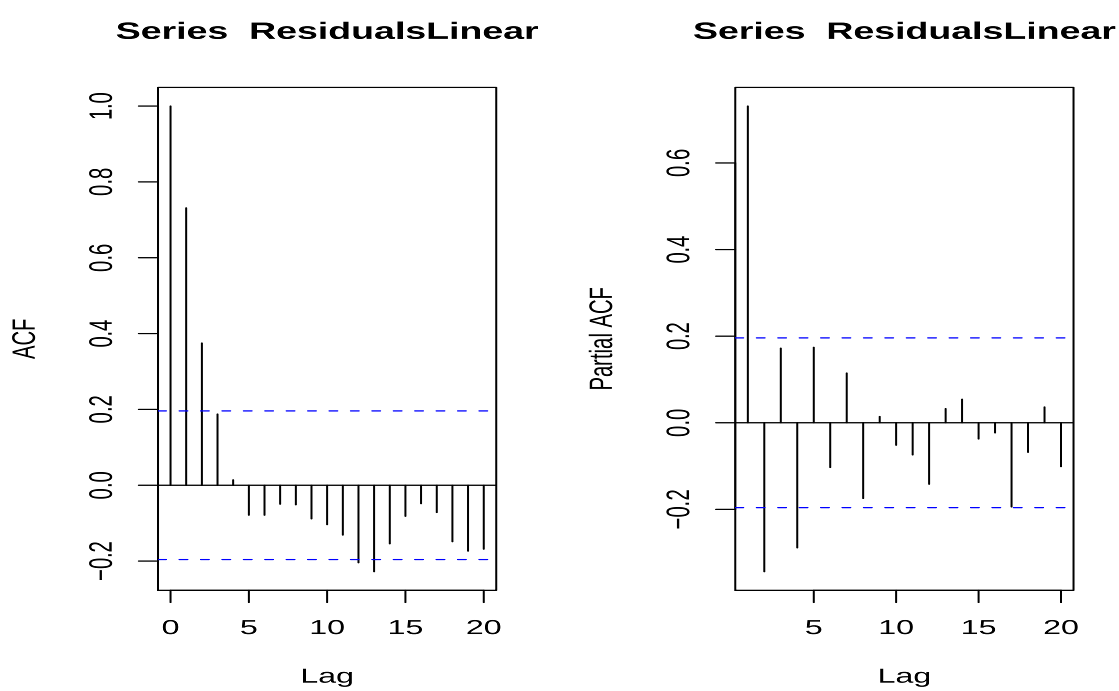
```
plot.ts(ResidualsExpSmooth);
```



The obtained pictures suggest that the residuals are stationary.

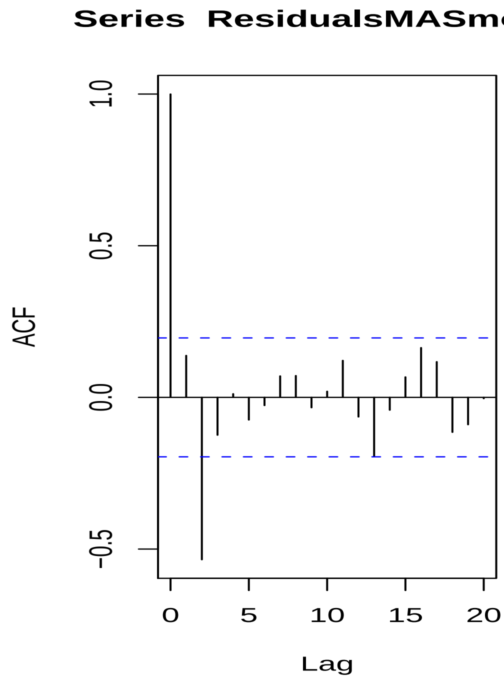
(g) In all three case we plot ACF and PACF

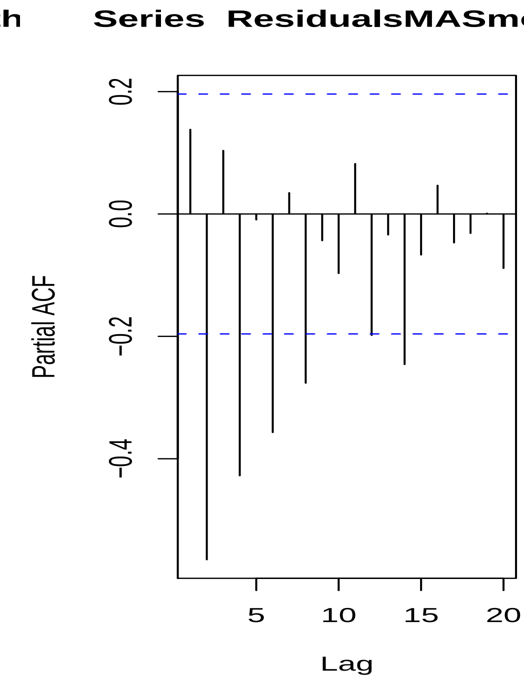
```
acf(ResidualsLinear); pacf(ResidualsLinear);
```



Note that for the parametric linear trend, ACF and PACF look similar to the original time series in (b).

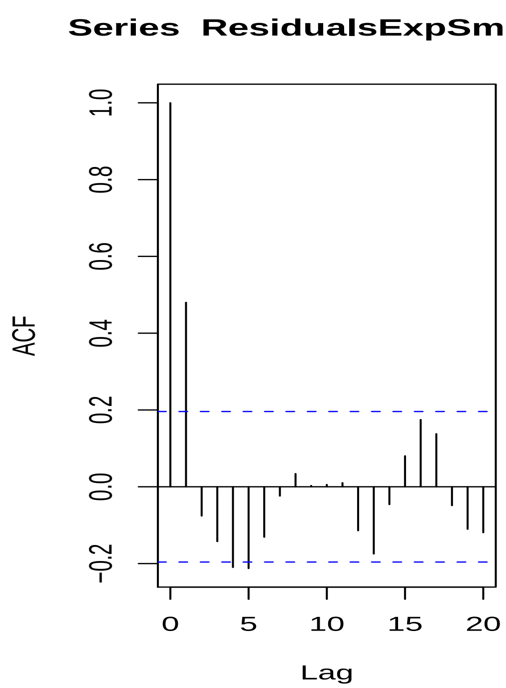
```
acf(ResidualsMASmooth); pacf(ResidualsMASmooth);
```

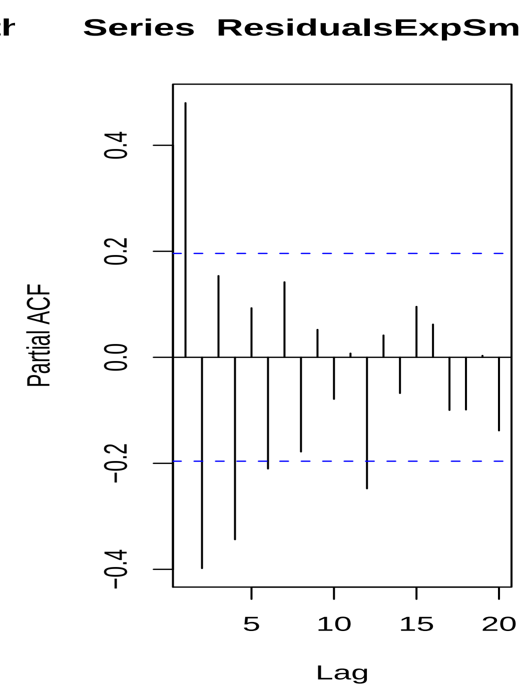




Note that for the moving average smooth, ACF and PACF look different as compared to the original time series in (b). It is hard to specify the model here. It looks like MA(1) model with some AR part. This is the price you pay when you apply nonparametric smooth.

```
acf(ResidualsExpSmooth); pacf(ResidualsExpSmooth);
```





Note that for the exponential smooth, ACF and PACF look different as compared to the original time series in (b). I would conclude that residuals follows MA(1) model. Thus, I would not specify the model correctly. This is the price you pay when you apply nonparametric smooth.

CONCLUSION: Whenever possible estimate trend in a parametric way (linear, polynomial).

Marking: 1 point for part (b) - only if you provide two plots and correctly identify a model based on your plots. Your identification may be different from the model you generated (as in my case - ARMA(1,1) instead of

ARMA(1,2), but you have to follow the graphs); 1 point for parametric trend - you have to write what function did you choose, what are your parameter estimates and plot a time series Y_t together with the trend; 1 point for moving average smoothing - you have plot a time series Y_t together with the trend; 1 point for exponential smoothing - you have plot a time series Y_t together with the trend; 1 point for three plots of residuals in part (f); 3 points for part (g) - 1 point for each set of residuals. Total: 7 points.