MAT3379 (Winter 2013)

Assignment 2

Due date (Assignment 2): 4 March 2013

FINAL VERSION

Q1. (Theoretical Question - 4 points). Yule-Walker procedure for AR(p) models. Assume that Z_t are i.i.d random variables with mean 0 and variance σ^2.

(a) Apply the Yule-Walker procedure to obtain $P_n X_{n+2}$ (two step prediction) for AR(1) model $X_t = \phi X_{t-1} + Z_t$, $|\phi| < 1$. Compute the corresponding $\mathrm{MSPE}_n(2)$. Can you guess a general formula for $P_n X_{n+k}$? (Hint: Look at Example 4.1 in Lecture Notes).

(b) Apply the Yule-Walker procedure to obtain $P_n X_{n+1}$ for AR(2) model $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$. Compute the corresponding $\mathrm{MSPE}_n(1)$.

Solution to Q1:

(a) In Example 4.1 we learned that for AR(1) model we have $P_n X_{n+1} = \phi X_n$. Now, we try to guess $P_n X_{n+2}$. If we happen to have observations $X_1, \ldots, X_{n+1}$, then prediction of the next X_{n+2}th value is ϕX_{n+1}. However, we have only n observations, so that in the latter formula we have to "predict" X_{n+1}. From Example 4.1, prediction of X_{n+1} has the form ϕX_n. Hence, we may guess that $P_n X_{n+2} = \phi(\phi X_n) = \phi^2 X_n$. We have to verify it. As in Example 4.1, recall that

$$\gamma_X(h) = \phi^h \frac{\sigma^2}{1-\phi^2}, \qquad h \geq 0.$$

Then

$$\gamma(n;k) = \gamma(n;2) = (\gamma_X(2), \ldots, \gamma_X(n+1))^T = \frac{\sigma^2}{1-\phi^2}(\phi^2, \ldots, \phi^{n+1})^T.$$

The general Yule-Walker equation for one-step prediction is

$$(1) \quad \begin{bmatrix} \gamma_X(0) & \gamma_X(1) & \gamma_X(2) & \gamma_X(3) & \cdots & \gamma_X(n-1) \\ \gamma_X(1) & \gamma_X(0) & \gamma_X(1) & \gamma_X(2) & \cdots & \gamma_X(n-2) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_X(n-1) & \gamma_X(n-2) & \gamma_X(n-3) & \gamma_X(n-4) & \cdots & \gamma_X(0) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \gamma_X(2) \\ \vdots \\ \gamma_X(n+1) \end{bmatrix}.$$

In the particular AR(1) case it becomes

$$\frac{\sigma^2}{1-\phi^2} \begin{bmatrix} 1 & \phi & \phi^2 & \phi^3 & \cdots & \phi^{n-1} \\ \phi & 1 & \phi & \phi^2 & \cdots & \phi^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi^{n-1} & \phi^{n-2} & \phi^{n-3} & \phi^{n-4} & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \frac{\sigma^2}{1-\phi^2} \begin{bmatrix} \phi^2 \\ \vdots \\ \phi^{n+1} \end{bmatrix}.$$

We may easily verify that our "guess" $(a_1, \ldots, a_n) = (\phi^2, 0, \ldots, 0)$ is the correct answer.
Furthermore,

$$\mathrm{MSPE}_n(2) = \gamma_X(0) - \underline{a}_n^T \gamma(n;2) = \frac{\sigma^2}{1-\phi^2} - \phi^2 \gamma_X(2) = \frac{\sigma^2}{1-\phi^2} - \frac{\sigma^2\phi^4}{1-\phi^2} = (1-\phi^4)\frac{\sigma^2}{1-\phi^2}\,.$$

We can also see that in a general case $P_n X_{n+k} = \phi^k X_n$ and $\mathrm{MSPE}_n(k) = (1-\phi^{2k})\dfrac{\sigma^2}{1-\phi^2}$.

(b) In Example 4.1 we learned that for AR(1) model $X_t = \phi X_{t-1} + Z_t$ the one step prediction is $\hat{X}_{n+1} = P_n X_{n+1} = \phi X_n$. Note that the defining formula for the true $(n+1)$st observation $X_{n+1} = \phi X_n + Z_{n+1}$ and the one-step prediction formula $\hat{X}_{n+1} = \phi X_n$ differ only by Z_{n+1}. (We can not use Z_{n+1} for prediction since we do not observe it!).
Now, AR(2) model is $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$. Hence, we may guess that the one-step prediction for AR(2) has the form $P_n X_{n+1} = \phi_1 X_n + \phi_2 X_{n-1}$, that is $(a_1, a_2, \ldots, a_n) =$. We verify it by checking validity

of the Yule-Walker equation for two-step prediction (see (9) in Lecture Notes):

$$\begin{bmatrix} \gamma_X(0) & \gamma_X(1) & \gamma_X(2) & \gamma_X(3) & \cdots & \gamma_X(n-1) \\ \gamma_X(1) & \gamma_X(0) & \gamma_X(1) & \gamma_X(2) & \cdots & \gamma_X(n-2) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_X(n-1) & \gamma_X(n-2) & \gamma_X(n-3) & \gamma_X(n-4) & \cdots & \gamma_X(0) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \gamma_X(1) \\ \vdots \\ \gamma_X(n) \end{bmatrix}.$$

With the chosen $(a_1, a_2, \ldots, a_n)$ we get zeros on both left and right hand side for all rows except the first and the second one. For the first and the second row we get, respectively,

$$\phi_1\gamma_X(0) + \phi_2\gamma_X(1) = \gamma_X(1);$$
$$\phi_1\gamma_X(1) + \phi_2\gamma_X(0) = \gamma_X(2).$$

Now, looking at Assignment 1, Question 4a), equations (1) and (6) with $h = 2$, we can recognize that the above equations are exactly formulas that are valid for covariances of AR(2) model. That is, we verified that our guess was correct.

Marking scheme for Q1:

4 points for part a). Correct reference to Yule-Walker procedure - 1 point; correct answer - 1 point; correct evaluation of $\mathrm{MSPE}_n(2)$; correct guess for a general k - 1 point. You do not need to provide the formula for $\mathrm{MSPE}_n(k)$. Part b) will not be marked.

Q2. (Theoretical Question - 4 points). Durbin-Levinson procedure and PACF for AR(p) models. Assume that Z_t are i.i.d random variables with mean 0 and variance σ^2.

Use Theorem 4.2 to obtain coefficients ϕ_{11}, ϕ_{22}, ϕ_{33} for AR(2) model. (Hint: cf. Example 4.3).

Solution to Q2:

Recall first from Assignment 1, Question 4a), equations (1), (5), (6) that for AR(2) we have the following formulas for the covariances:

$$\gamma_X(h) = \phi_1\gamma_X(h) + \phi_2\gamma_X(h-2)\ , \qquad h \geq 2\ ,$$
$$\gamma_X(1) = \gamma_X(0)\frac{\phi_1}{1-\phi_2}$$

and $\gamma_X(0)$ solves (5) in Assignment 1, Question 4a).

From Theorem 4.2 in Lecture Notes we have:

$$\phi_{11} = \rho_X(1) = \frac{\gamma_X(1)}{\gamma_X(0)} = \frac{\phi_1}{1-\phi_2}.$$

Furthermore,

$$\phi_{22} = [\gamma_X(2) - \phi_{11}\gamma_X(1)]\,/v_1^{-1}. \tag{2}$$

Recall from Assignment 1, Question 4a) that the formula for covariances of AR(2) is

$$\phi_1\gamma_X(h-1) + \phi_2\gamma_X(h-2) = \gamma_X(h)\ , \qquad h \geq 2, \tag{3}$$
$$\gamma_X(1) = \gamma_X(0)\frac{\phi_1}{1-\phi_2}.$$

Use the above formulas (first one with $h = 2$) and replace $\gamma_X(2)$ in (2) to get

$$\phi_{22} = [\phi_1\gamma_X(1) + \phi_2\gamma_X(0) - \phi_{11}\gamma_X(1)]\,/v_1^{-1} \tag{4}$$
$$= \left[\gamma_X(0)\frac{\phi_1^2}{1-\phi_2} + \phi_2\gamma_X(0) - \gamma_X(0)\frac{\phi_1^2}{(1-\phi_2)^2}\right]/v_1^{-1}. \tag{5}$$

Now, from Theorem 4.2,

$$v_1 = v_0(1-\phi_{11}^2) = \gamma_X(0)\left(1 - \frac{\phi_1^2}{(1-\phi_2)^2}\right). \tag{6}$$

If you combine (4)-(6) together you will get

$$\phi_{22} = \phi_2. \tag{7}$$

Now, for ϕ_{33}: recall that this value represents partial autocovariance at lag 3. It was mentioned in class (see also Lecture Notes, Example 4.4) that for AR(2), PACF vanishes after lag 2. Hence, we should get $\phi_{33} = 0$. We will verify it. From Theorem 4.2 we get

$$\phi_{33} = \left[\gamma_X(3) - \phi_{21}\gamma_X(2) - \phi_{22}\gamma_X(1)\right]/v_2^{-1}.$$

Use (3) with $h = 3$ to get

$$\phi_{33} = \left[(\phi_1 - \phi_{21})\gamma_X(2) - (\phi_2 - \phi_{22})\gamma_X(1)\right]/v_2^{-1}.$$

Keeping in mind (7), in order to show that $\phi_{33} = 0$ it is enough to show that $\phi_{21} = \phi_1$. We use again Theorem 4.2:

$$\phi_{21} = \phi_{11} - \phi_{22}\phi_{11} = \phi_{11}(1 - \phi_2 2) = \frac{\phi_1}{1-\phi_2}(1-\phi_2) = \phi_1.$$

That is, $\phi_{33} = 0$.
Marking: 1 point for correct value of ϕ_{11}, 2 points for the correct value of ϕ_{22} and 1 point for the correct value of ϕ_{33}. Total: 4 points.

Q3. (Theoretical Question - 5 points). Yule-Walker procedure and Durbin-Levinson algorithm for MA(q) models.

Consider the MA(1) model $X_t = Z_t + \theta Z_{t-1}$, $\theta \in \mathbb{R}$, where Z_t are i.i.d. random variables with mean 0 and variance σ^2. Our goal is to find the best linear predictor $P_n X_{n+1}$ of X_{n+1} based on $X_1, \ldots, X_n$.

(a) Let $n = 1$. Use the formula $\Gamma_n \mathbf{a}_n = \gamma(n; 1)$ to conclude that

$$P_1 X_2 = \frac{\gamma_X(1)}{\gamma_X(0)} X_1 = \frac{\theta}{1+\theta^2} X_1.$$

(b) Let $n = 2$. Use the formula $\Gamma_n \mathbf{a}_n = \gamma(n; 1)$ to obtain coefficients a_1, a_2 in $P_2 X_3 = a_1 X_2 + a_2 X_1$.

(c) Let, as before, $n = 2$. Apply the Durbin-Levinson algorithm to get $P_2 X_3 = \phi_{21} X_2 + \phi_{22} X_1$. You should get the same answer as in the previous part; note however that there is a change of notation: (a_1, a_2) are changed into (ϕ_{21}, ϕ_{22}).

Solution to Q3:

(a) Since $n = 1$, we have $\Gamma_1 = \gamma_X(0)$, $a_n = a_1$, $\gamma(n; 1) = \gamma_X(1)$, we obtain immediately $P_1 X_2 = \dfrac{\gamma_X(1)}{\gamma_X(0)} X_1$.
Now, the covariances for MA(1) are given in Lecture Notes, Section 2.1, Example 4. This leads to

$$P_1 X_2 = \frac{\gamma_X(1)}{\gamma_X(0)} X_1 = \frac{\theta}{1+\theta^2} X_1.$$

(b) Since $n = 2$ we have to solve

$$\mathbf{a}_2 = \Gamma_2^{-1}\gamma(2; 1),$$

that is

$$(a_1, a_2)^T = \begin{bmatrix} \gamma_X(0) & \gamma_X(1) \\ \gamma_X(1) & \gamma_X(0) \end{bmatrix}^{-1} \begin{pmatrix} \gamma_X(1) \\ \gamma_X(2) \end{pmatrix}.$$

Hint: work with general covariances $\gamma_X(k)$, do not plug-in specific values yet. You can also solve

$$\gamma_X(0)a_1 + \gamma_X(1)a_2 = \gamma_X(1), \tag{8}$$
$$\gamma_X(1)a_1 + \gamma_X(0)a_2 = \gamma_X(2). \tag{9}$$

For MA(1) we know that $\gamma_X(2) = 0$, hence the second equation yields

$$a_2 = -\frac{\gamma_X(1)}{\gamma_X(0)} a_1.$$

Plug-in this into the first equation to obtain

$$a_1 = \frac{\gamma_X(0)\gamma_X(1)}{\gamma_X^2(0) - \gamma_X^2(1)}.$$

Now, we plug-in $\gamma_X(1) = \sigma^2\theta$, $\gamma_X(2) = 0$ to obtain

$$a_1 = -\frac{\theta(1+\theta^2)}{\theta^2 - (1+\theta^2)^2}$$
$$a_2 = -\frac{\theta}{1+\theta^2}\frac{\theta(1+\theta^2)}{\theta^2-(1+\theta^2)^2} = -\frac{\theta^2}{(1+\theta^2)^2-\theta^2}.$$

Notes: To do two-step prediction P_1X_3 you would solve

$$\gamma_X(0)a_1 + \gamma_X(1)a_2 = \gamma_X(2),$$
$$\gamma_X(1)a_1 + \gamma_X(0)a_2 = \gamma_X(3).$$

Also, not that unlike in AR(p) case, it would be very difficult to obtain formulas P_nX_{n+1} for general n.

(c) We apply directly Theorem 4.2. We have $\phi_{11} = \gamma_X(1)/\gamma_X(0) = \theta/(1+\theta^2)$. Next, $v_0 = \gamma_X(0) = \sigma^2(1+\theta^2)$,

$$v_1 = v_0[1-\phi_{11}^2] = \sigma^2\frac{(\theta^2+1)^2-\theta^2}{\theta^2+1}.$$

Thus,

$$\phi_{22} = v_1^{-1}[\gamma_X(2) - \phi_{11}\gamma_X(1)] = -v_1^{-1}\phi_{11}\gamma_X(1) = -v_1^{-1}\frac{\sigma^2\theta^2}{\theta^2+1} = -\frac{\theta^2}{(\theta^2+1)^2-\theta^2}$$

Finally,

$$\phi_{21} = \phi_{11} - \phi_{22}\phi_{11} = -\frac{\theta(1+\theta^2)}{\theta^2-(1+\theta^2)^2}.$$

Note that the Durbin-Levinson and Yule-Walker lead to the same results. It should be like that since we solve the same prediction problem, only methods are different.

Marking: Part a): 1 point for the correct answer. Part b): 1 point for each correct a_1 and a_2. Part c): 1 point for each correct ϕ_{21} and ϕ_{22}.

Q4. (Theoretical Question). Yule-Walker prediction for ARMA(p,q) models.

Consider the ARMA(1, 1) model $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$, $|\phi| < 1$, $\theta \in \mathbb{R}$, where Z_t are i.i.d. random variables with mean 0 and variance σ^2. Our goal is to find the best linear predictor P_nX_{n+1} of X_{n+1} based on $X_1, \ldots, X_n$.

- Let $n = 1$. Use the formula $\Gamma_n\mathbf{a}_n = \gamma(n; 1)$ to obtain a_1 in $P_1X_2 = a_1X_1$.
- Let $n = 2$. Use the formula $\Gamma_n\mathbf{a}_n = \gamma(n; 1)$ to obtain coefficients a_1, a_2 in $P_2X_3 = a_1X_2 + a_2X_1$.

Hint: We have the following formulas for the covariance function:

$$\gamma_X(0) = \sigma^2\left[1 + \frac{(\phi+\theta)^2}{1-\phi^2}\right],$$
$$\gamma_X(1) = \sigma^2\left[(\phi+\theta) + \frac{(\phi+\theta)^2\phi}{1-\phi^2}\right],$$
$$\gamma_X(h) = \phi^{h-1}\gamma_X(1), \qquad h \geq 2. \tag{10}$$

Solution to Q4:

Solution mirrors Question 3, parts a) and b).

For $n = 1$ we use the formula $\Gamma_n\mathbf{a}_n = \gamma(n; 1)$ together with the covariances given above to conclude that

$$P_1X_2 = \frac{\gamma_X(1)}{\gamma_X(0)}X_1 = \frac{\phi\theta(1+\phi+\theta)}{1+2\phi\theta+\theta^2}X_1.$$

For $n = 2$ we start with equations (8)-(9). The only difference between MA(1) considered before and ARMA(1,1) considered now, is that we have different covariances. In particular, using (10) with $h = 1$ we re-write (8)-(9) as

$$\gamma_X(0)a_1 + \gamma_X(1)a_2 = \gamma_X(1), \tag{11}$$
$$\gamma_X(1)a_1 + \gamma_X(0)a_2 = \phi\gamma_X(1). \tag{12}$$

From the first equation we get

$$a_2 = 1 - \frac{\gamma_X(1)}{\gamma_X(0)} a_1.$$

Plug-in into the second one and solve for a_1 and a_2. The formulas will be messy, but they depend explicitly on $\gamma_X(0)$ and $\gamma_X(1)$.
Marking: This question will not be marked.

Q5. (Theoretical Question - 4 points). Yule-Walker estimation for AR(p) models. Assume that Z_t are i.i.d random variables with mean 0 and variance σ^2.

Consider AR(2) model $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$. Use Theorem 5.2 to derive confidence intervals for $\hat{\phi}_1$ and $\hat{\phi}_2$.

Solution to Q5:

Theorem 5.2 for $p = 2$ tells us that

$$(\hat{\phi}_1, \hat{\phi}_2)^T \sim N\left((\phi_1, \phi_2), \frac{1}{n}\sigma^2 \Gamma_2^{-1}\right).$$

Note that $\sigma^2 \Gamma_2^{-1}$ is a 2×2 matrix, called limiting variance-covariance matrix. The diagonal entries correspond to asymptotic variance of corresponding estimators, for example, the first diagonal entry is the asymptotic variance of $\mathrm{Var}(\hat{\phi}_1)$ that is used to construct confidence interval.

The limiting variance-covariance matrix is given in Example 5.4 (Lecture Notes):

$$\sigma^2 \Gamma_2^{-1} = \begin{bmatrix} 1 - \phi_2^2 & -\phi_1(1+\phi_2) \\ -\phi_1(1+\phi_2) & 1 - \phi_1^2 \end{bmatrix}.$$

Hence, the theoretical confidence interval for ϕ_1 and ϕ_2 is, respectively,

$$\hat{\phi}_1 \pm z_{\alpha/2} \frac{1}{\sqrt{n}} \sqrt{1 - \phi_2^2}, \qquad \hat{\phi}_2 \pm z_{\alpha/2} \frac{1}{\sqrt{n}} \sqrt{1 - \phi_1^2}. \tag{13}$$

These confidence intervals are not practical since ϕ_1, ϕ_2 are unknown. Practical confidence intervals are

$$\hat{\phi}_1 \pm z_{\alpha/2} \frac{1}{\sqrt{n}} \sqrt{1 - \hat{\phi}_2^2}, \qquad \hat{\phi}_2 \pm z_{\alpha/2} \frac{1}{\sqrt{n}} \sqrt{1 - \hat{\phi}_1^2}.$$

Now, we have to give formulas for $\hat{\phi}_1$ and $\hat{\phi}_2$. Use equation (24) of Lecture Notes to obtain Yule-Walker equations

$$(\phi_1, \phi_2)^T = \Gamma_2^{-1} (\gamma_X(1), \gamma_X(2))^T = \frac{1}{\gamma_X^2(1) - \gamma_X^2(0)} \begin{pmatrix} \gamma_X(0)\gamma_X(1) - \gamma_X(1)\gamma_X(2) \\ -\gamma_X(1)\gamma_X(1) + \gamma_X(0)\gamma_X(2) \end{pmatrix}.$$

Replacing $\gamma_X(k)$ with their estimated values we obtain e.g.

$$\hat{\phi}_1 = \frac{\hat{\gamma}_X(0)\hat{\gamma}_X(1) - \hat{\gamma}_X(1)\hat{\gamma}_X(2)}{\hat{\gamma}_X^2(1) - \hat{\gamma}_X^2(0)}. \tag{14}$$

Marking: 2 points (13) - one for each confidence interval. 1 point for some steps towards $\hat{\phi}_1$ and $\hat{\phi}_2$. One point for (14), that is at least one correct value for $\hat{\phi}_j$, $j = 1, 2$. Total: 4 points.

Q6. (Theoretical Question). This question has been moved to Assignment 3. If you included a solution in Assignment 2, please include it again in Assignment 3.

Maximum Likelihood Estimation for AR(p) models.

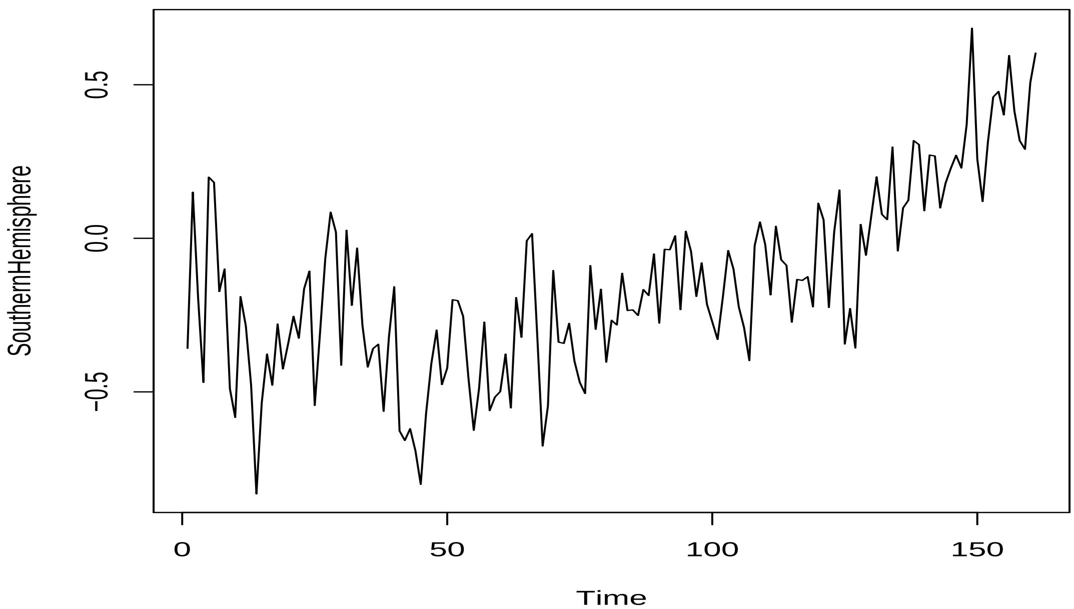
Consider AR(1) model $X_t = \phi X_{t-1} + Z_t$, where Z_t are i.i.d. normal random variables with mean zero and variance σ^2. Derive MLE for ϕ and σ^2. (Hint: You should get formulas as in Example 6.2, but I need to see calculations).

Q7. (Practical Question). You may use codes available on the webpage.

- Download a data set.
- Remove trend using any of the methods, if needed. You should obtain stationary residuals. State your chosen $\hat{m}_t$.
- Plot the original sequence together with the estimated trend (1 graph).
- Plot residuals, then ACF and PACF (3 graphs).

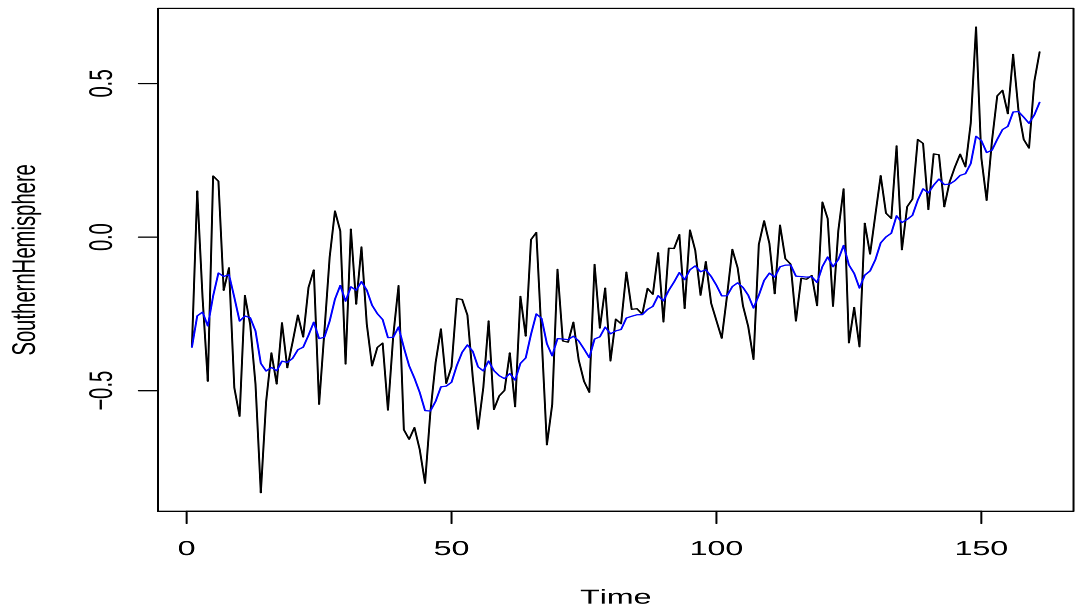
Solution to Q7:

- I consider `SouthernHemisphere.txt`. I typed
  `plot.ts(SouthernHemisphere)`
  and obtained the following graph:

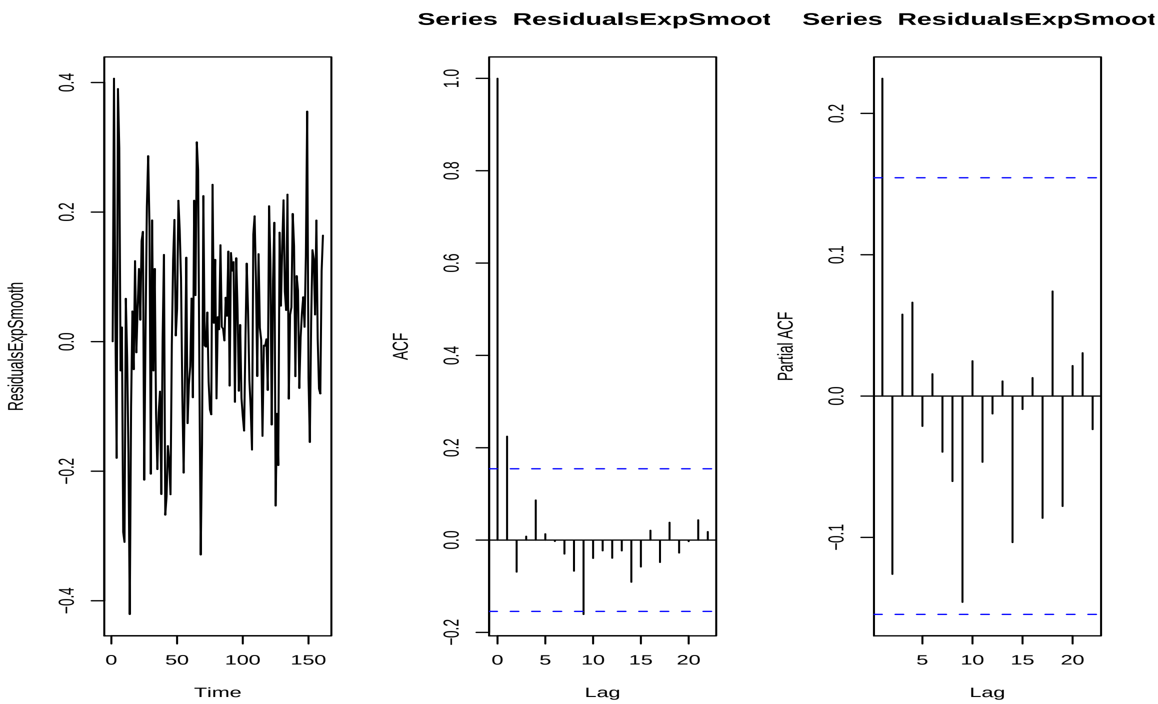


- The graph suggests that there is some trend. Linear trend does not seem to be appropriate. I applied exponential smoothing

```
SmoothedTimeSeries=ExpSmooth(SouthernHemisphere,0.2);
n=length(SouthernHemisphere); Time=c(1:n);
plot(Time,SouthernHemisphere,type="l",col="black");
points(Time,SmoothedTimeSeries,type="l",col="blue");
ResidualsExpSmooth=SouthernHemisphere-SmoothedTimeSeries;
```



- I plot residuals, ACF, PACF.
`plot.ts(ResidualsExpSmooth); acf(ResidualsExpSmooth);`
`pacf(ResidualsExpSmooth);`



PACF excludes MA component; suggests AR(1). ACF suggests that it is either AR model of small order or MA(1). Combining together; I choose AR(1) model.

Marking: This part will not be marked.

Q8. (Theoretical/Practical Question - 7 points) In this question we develop Yule-Walker estimator in ARMA$(1,1)$ model and study its numerical performance.

- Recall from lectures that in AR(1) model $X_t = \phi X_{t-1} + Z_t$ the Yule-Walker estimator is

$$\hat{\phi} = \frac{\hat{\gamma}_X(1)}{\hat{\gamma}_X(0)} = \hat{\rho}_X(1) , \qquad \hat{\sigma}^2 = \hat{\gamma}_X(0) - \hat{\phi}\hat{\gamma}_X(1) = \hat{\gamma}_X(0) - \hat{\rho}_X(1)^2\hat{\gamma}_X(0).$$

(a) Numerical experiment for AR(1):
  - Load into `R` the file Data-AR.txt. (Just type Data=scan() and then copy and paste). This is data set generated from AR(1) model with $\phi = 0.8$.
  - Type `var(Data)` to obtain $\hat{\gamma}_X(0)$.
  - Type `ACF<-acf(Data)`. Then type `ACF`. You will get $\hat{\rho}_X(h)$, the estimators of $\rho_X(h)$. The second entry will be $\hat{\rho}_X(1)$. Via the formula above this is also $\hat{\phi}$.
  - Write the final values for $\hat{\phi}$ and $\hat{\sigma}^2$.
  - Compare your estimated $\hat{\phi}$ with the true ϕ.

(b) Consider ARMA$(1,1)$ model $X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}$, $|\phi| < 1$, so that the sequence X_t is causal. Apply the Yule-Walker procedure to get the estimators for ϕ, θ and $\sigma^2 = \mathrm{Var}(Z_t)$.

HINT: You should get

$$\phi = \frac{\gamma_X(2)}{\gamma_X(1)} , \quad \gamma_X(1) = \phi\gamma_X(0) + \theta\sigma^2 , \quad \gamma_X(0) = \sigma^2\left[1 + \frac{(\theta+\phi)^2}{1-\phi^2}\right].$$

(c) Numerical experiment for ARMA$(1,1)$:
  - Load into `R` the file Data-ARMA.txt. (Just type Data=scan() and then copy and paste). This is data set generated from ARMA$(1,1)$ model with $\phi = 0.8$ and $\theta = 1$.
  - Write the final values for $\hat{\phi}$, $\hat{\theta}$ and $\hat{\sigma}^2$.
  - Compare your estimated $\hat{\phi}$ with the true ϕ. Which estimate is more accurate, for ARMA$(1,1)$ or for AR(1)?

Solution to Q8:

(a) After typing `var(Data)` we obtain $\hat{\gamma}_X(0) = 2.295053$. After typing `ACF<-acf(Data)` and `ACF` we get

```
    0     1     2     3     4     5     6     7     8     9    10    11    12
1.000 0.747 0.550 0.429 0.333 0.279 0.205 0.085 0.024 0.009 0.021 0.084 0.080
   13    14    15    16    17    18    19    20
0.080 0.095 0.092 0.082 0.071 0.056 0.030 0.058
```

The second entry is $\hat{\rho}_X(1) = 0.747$. Thus, $\hat{\phi} = 0.747$. Recall that the true parameter was $\phi = 0.8$. Furthermore,

$$\hat{\sigma}^2 = 2.295053 - 0.747^2 * 2.295053 = 1.014.$$

(b) For ARMA(1,1) we have

$$\gamma_X(h) = \phi^{h-1}\gamma_X(1), \qquad h \geq 2;$$
$$\gamma_X(1) = \sigma^2\left[(\theta+\phi) + \frac{\phi(\theta+\phi)^2}{1-\phi^2}\right]$$
$$\gamma_X(0) = \sigma^2\left[1 + \frac{(\theta+\phi)^2}{1-\phi^2}\right].$$

Clearly, $\gamma_X(2)/\gamma_X(1) = \phi$ that gives the first equation. The third equation is for free. Also, we immediately the second equation by applying the above formulas for $\gamma_X(1)$ and $\gamma_X(0)$.

(c) Now, numerical experiment for ARMA$(1,1)$. After typing `ACF<-acf(Data)` and `ACF` we get

```
    0      1      2      3      4      5      6      7      8      9     10
 1.000  0.509  0.204  0.122  0.061  0.015  0.083  0.059  0.049 -0.023  0.025
    11     12     13     14     15     16     17     18     19     20
 0.063  0.074  0.047 -0.095 -0.208 -0.173 -0.179 -0.180 -0.144 -0.051
```

From this we read: $\hat{\rho}_X(1) = 0.509$; $\hat{\rho}_X(2) = 0.204$. Now, from the formula above

$$\hat{\phi} = \frac{\hat{\gamma}_X(2)}{\hat{\gamma}_X(1)} = \frac{\hat{\rho}_X(2)}{\hat{\rho}_X(1)} = 0.204/0.509 = 0.4.$$

Type `var(Data)` to get $\hat{\gamma}_X(0) = 1.496668$. From this we get $\hat{\gamma}_X(1) = \hat{\rho}_X(1)\hat{\gamma}_X(0) = 0.509 * 1.496668 = 0.7618$ We take the system of two equations obtained from Yule-Walker procedure and we replace values with their estimators

$$\hat{\gamma}_X(1) = \hat{\phi}\hat{\gamma}_X(0) + \hat{\theta}\hat{\sigma}^2 , \quad \hat{\gamma}_X(0) = \hat{\sigma}^2 \left[1 + \frac{(\hat{\theta}+\hat{\phi})^2}{1-\hat{\phi}^2} \right] .$$

We obtain

$$0.7618 = \hat{\gamma}_X(1) = \hat{\phi}\hat{\gamma}_X(0) + \hat{\theta}\hat{\sigma}^2 = 0.4 * 1.496668 + \hat{\theta}\hat{\sigma}^2$$

and

$$1.496668 = \hat{\gamma}_X(0) = \hat{\sigma}^2 \left[1 + \frac{(\hat{\theta}+\hat{\phi})^2}{1-\hat{\phi}^2} \right] = \hat{\sigma}^2 \left[1 + \frac{(\hat{\theta}+0.4)^2}{1-0.4^2} \right] .$$

The second equation has two solutions, take the one which is smaller than 1:

$$\hat{\phi} = 0.4; \hat{\theta} = 0.14 \, , \hat{\sigma}^2 = 1.42.$$

Note that the true parameters were $\phi = 0.8$; $\theta = 1$; $\sigma = 1$. Hence, the fact that MA part is present messes up estimation of the autoregressive coefficient ϕ.

Marking: Part (a) - 1 point for each correct value of $\hat{\phi}$ and $\hat{\sigma}^2$. Total 2 points for part (a); Part (c): 1 point each for each correct value of $\hat{\phi}$, $\hat{\theta}$ and $\hat{\sigma}^2$. 1 point for a comment on accuracy of estimation of the AR coefficient. Total 4 points for part c). Part b) - 1 point for making a link between formulas for covariances and formulas that appear in the HINT. Total: 7 points.

Q9. (Theoretical-Practical Question - 3 points).

(a) One hundred observations from AR(1) yields the following sample statistics:

$$\bar{x} = 0 \, , \quad \hat{\gamma}_X(0) = 1.1 \, , \quad \hat{\rho}_X(1) = 0.42.$$

- Find the Yule-Walker estimators of ϕ and σ^2.
- Write the confidence interval for ϕ.
- If $X_{100} = 1.5$, what is the predicted value of X_{101}? What is the squared error of this prediction?

(b) Two hundred observations from AR(2) yields the following sample statistics:

$$\bar{x} = 3.82 \, , \quad \hat{\gamma}_X(0) = 1.15 \, , \quad \hat{\rho}_X(1) = 0.427 \, , \hat{\rho}_2 = 0.475.$$

- Find the Yule-Walker estimators of ϕ_1, ϕ_2 and σ^2.
- Is the estimated model causal?.
- If $X_{100} = 3.84$ and $X_{99} = 3.26$, what is the predicted value of X_{101}?

Solution to Q9:

(a) We have the following formulas for Yule-Walker estimators in AR(1) case:

$$\hat{\phi} = \hat{\rho}_X(1) = \hat{\rho}_X(1) = 0.42 \, ,$$

$$\hat{\sigma}^2 = \hat{\gamma}_X(0) - \hat{\phi}\hat{\gamma}_X(1) = \hat{\gamma}_X(0) - \hat{\phi}^2\hat{\gamma}_X(0) = 1.1 - 0.42^2 * 1.1 = 0.90596.$$

The confidence interval for ϕ is

$$\hat{\phi} \pm z_{\alpha/2}\frac{1}{\sqrt{n}}\hat{\sigma}\frac{1}{\sqrt{\hat{\gamma}_X(0)}}.$$

Choose $\alpha = 0.05$; then $z_{0.025} = 1.96$. Thus

$$0.42 \pm 1.96 * \frac{1}{\sqrt{100}} * 0.90596 * \frac{1}{\sqrt{1.1}} = 0.42 \pm 0.17.$$

The prediction for the 101st is $\hat{\phi} \times X_{100} = 0.63$. The squared error of the prediction is

$$\hat{\sigma}^2(1 + 1/n) = 0.90596 * (1 + 1/100).$$

(b) From the data we compute $\hat{\gamma}_X(1) = \hat{\rho}_X(1) \times \hat{\gamma}_X(0) = 0.49$, $\hat{\gamma}_X(2) = \hat{\rho}_2 \times \hat{\gamma}_X(0) = 0.55$. We have to solve

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \Gamma_2^{-1} \begin{pmatrix} \hat{\gamma}_X(1) \\ \hat{\gamma}_X(2) \end{pmatrix}$$

where

$$\Gamma_2 = \begin{bmatrix} \hat{\gamma}_X(0) & \gamma_X(1) \\ \hat{\gamma}_X(1) & \gamma_X(0) \end{bmatrix}$$

We obtain

$$\begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} = \begin{pmatrix} 0.27 \\ 0.36 \end{pmatrix}$$

The predicted value is

$$\hat{X}_{101} - 3.82 = \hat{\phi}_1(X_{100} - 3.82) + \hat{\phi}_2(X_{99} - 3.82) = 0.27 * (3.84 - 3.82) + 0.36 * (3.26 - 3.82) = -0.1962.$$

Thus

$$\hat{X}_{101} = 3.82 - 0.1962 = 3.6238.$$

The autoregressive polynomial is

$$\phi(z) = 1 - 0.27z - 0.36z^2.$$

It has two solutions: $z_1 = 1.33$ and $z_2 = -2.0833$. Both have absolut values bigger than one, thus the model is causal and stationary.

Marking: 3 points for part (a), one for each subpart. Part (b) will not be marked.