4. (a) $\mathcal{L} = u(c,\ell) + \lambda\left[w(h-\ell) - \pi - T - c\right]$

$$u_c - \lambda = 0$$

$$u_\ell - \lambda = 0$$

$$w(h-\ell) + \pi - T - C = 0$$

(b) $u_{cc}\cdot dc + u_{c\ell}\cdot d\ell - d\lambda = 0$

$$u_{\ell c}\cdot dc + u_{\ell\ell}\cdot d\ell - w\,d\lambda - \lambda\,dw = 0$$

$$(h-\ell)\,dw - w\cdot d\ell + d\pi - dT - dc = 0$$

$$\begin{bmatrix} u_{cc} & u_{c\ell} & -1 \\ u_{\ell c} & u_{\ell\ell} & -w \\ -1 & -w & 0 \end{bmatrix}\begin{bmatrix} dc \\ d\ell \\ d\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda\,dw \\ -(h-\ell)dw - d\pi + dT \end{bmatrix}$$

(c) "A" matrix looks like Bordered Hessian for problem, except that border-terms are negative. In fact, it is the Bordered Hessian $\Rightarrow$ can be

$\Rightarrow$ the bordered Hessian can be written either way $\Rightarrow$ determinant unaffected.

(d) $|A| = (-1)\left[-w \cdot u_{ce} + u_{ee}\right] + w\left[-w \cdot u_{cc} + u_{ec}\right]$

$= -u_{ee} + 2w \cdot u_{ce} - w^2 u_{cc}$

(e) $|\bar{H}| = |A|$ from above

(f) For this problem, since $u(\cdot)$ is s.q.c., then $\bar{H}$ is negative definite, which in this case implies $|\bar{H}| > 0$

(g) $$\frac{\partial c}{\partial d\pi} = \frac{\begin{vmatrix} 0 & u_{ce} & -1 \\ 0 & u_{ee} & -w \\ -1 & -w & 0 \end{vmatrix}}{|A|}$$

$$= \frac{-u_{ee} + w \cdot u_{ce}}{|A|}$$

$$\frac{\partial e}{\partial \pi} = \frac{\begin{vmatrix} u_{cc} & 0 & -1 \\ u_{ec} & 0 & -w \\ -1 & -1 & 0 \end{vmatrix}}{|A|} = \frac{u_{ec} - w \cdot u_{cc}}{|A|}$$

(h) The assumption that c & ℓ are normal means that the demand for c & ℓ $\uparrow$'s as income $\uparrow$'s. The demand for c & ℓ is given by c^* & ℓ^*, & a Δ in income here is equivalent to $d\pi$, therefore, normality implies that

$$\frac{\partial c}{\partial \pi} > 0 \; ; \; \frac{\partial \ell}{\partial \pi} > 0$$

(5)

(a) Let $m(\sigma) = [c\, v(e)]^{1-\sigma} - 1$; $n(\sigma) = 1-\sigma$

$$m'(\sigma) = \ln[c \cdot v(e)]\,[c\, v(e)]^{1-\sigma}(-1)$$

$$n'(\sigma) = -1$$

$$m'(\sigma)/n'(\sigma) = \ln[c\, v(e)]\,[c\, v(e)]^{1-\sigma}$$
$$= [\ln c + \ln v(e)] \cdot [c\, v(e)]^{1-\sigma}$$

$$\lim_{\sigma \to 1} \frac{m'(\sigma)}{n'(\sigma)}$$
$$= [\ln c + \ln v(e)][c\, v(e)]^{0}$$
$$= \ln c + \ln v(e)$$

(b) from class & Williamson,

$$\frac{\partial \ell}{\partial w} = \frac{-u_c + (h-\ell)(u_{c\ell} - w\,u_{cc})}{\nabla} \qquad (*)$$

$$u_c = [c \cdot v(\ell)]^{-\sigma} \cdot v(\ell)$$

$$u_{cc} = -\sigma \cdot [c\, v(\ell)]^{-\sigma-1} (v(\ell))^2$$

$$u_{c\ell} = -v(\ell) \cdot \sigma \cdot [c\, v(\ell)]^{-\sigma-1} c\, v'(\ell)$$

$$= -\sigma [c \cdot v(\ell)]^{-\sigma} v'(\ell) + [c\, v(\ell)]^{-\sigma} v'(\ell)$$

$$= (1-\sigma)[c\, v(\ell)]^{-\sigma} v'(\ell)$$

(don't need to simplify more than this for part (c), but....

- from FOC's, we know that

$$w = u_\ell / u_c$$

$$= \frac{[\cdot]^{-\sigma} \cdot c\, v'(\ell)}{[\cdot]^{-\sigma} \cdot v(\ell)} = \frac{c \cdot v'(\ell)}{v(\ell)} \quad (*A)$$

So $w \cdot u_{cc} =$

$$c \frac{\cdot v'(\ell)}{v(\ell)} (-\sigma) [\cdot]^{-\sigma-1} \cdot (v(\ell))^2$$

$$= -\sigma [c\, v(\ell)]^{-\sigma} \cdot v'(\ell)$$

So $u_{c\ell} - w \cdot u_{cc}$

$$= (1-\sigma) [\cdot]^{-\sigma} v'(\ell) + \sigma [\cdot]^{-\sigma} v'(\ell)\, v'(\ell)$$

$$= [c\, v(\ell)]^{-\sigma} \cdot v'(\ell)$$

So can simplify $\frac{\partial \ell}{\partial w}$ as

$$\frac{\partial \ell}{\partial w} = -\frac{[c\, v(\ell)]^{-\sigma}}{\nabla} v(\ell) + \frac{(h-\ell)}{\nabla} [c\, v(\ell)]^{-\sigma} v'(\ell) \quad (*{*}A)$$

(c) Now in eq$^{\underline{m}}$, if $c = w(h-\ell)$:

find expression for $h-\ell$

$$h-\ell = \frac{c}{w} = \frac{c}{c \cdot w'(\ell)/v(\ell)} = v(\ell)/w'(\ell)$$

Sub into (* * *)

$$\frac{\partial \ell}{\partial w} = \frac{-[\cdot]^{-\sigma} v(\ell) + v(\ell)/w'(\ell) [\cdot]^{-\sigma} w'(\ell)}{\nabla}$$

$$= 0$$

$$\boxed{\frac{\partial \ell}{\partial w} = 0}$$

(d) We can interpret this result of zero as the substitution effect cancelling out the income effect - ie neither dominates the other.

(e) Here we can either think about the single period being "a long time"- ie many years - or alternatively think about requiring that these equations must hold every period. Since the SE cancels the IE, that agent's optimal choice of leisure doesn't change as the real wage changes. We can think about this as a vertical labour supply curve on a plot of wage on the vertical axes and hours on the horizontal axes . As the wages rises over time due to say a shifting labour demand curve, the real wage will rise, yet hours-worked will remain constant, consistent with the evidence given in the question.