ECON 4020 A Mid-Term Examination

Tuesday, October 16, 2012

Prof. K. G. Armstrong Name: ANSWER KEY

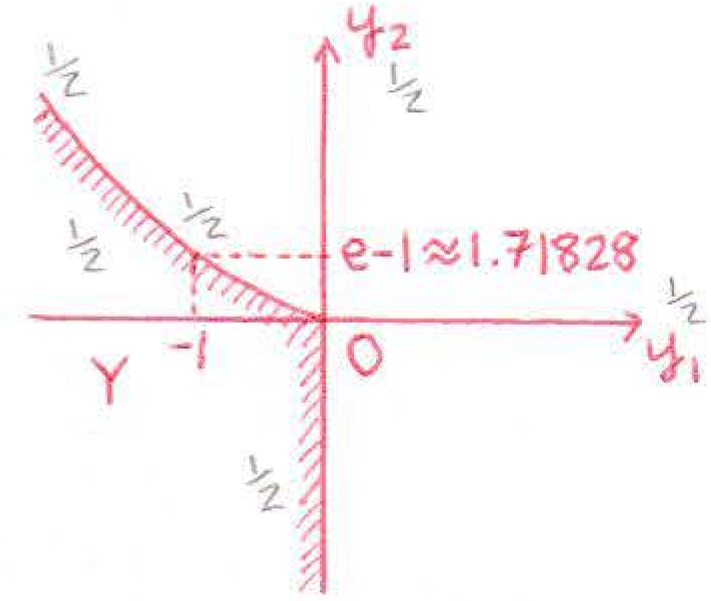
Instructions: Answer all questions in the space provided. **30 pts**

1. Consider the production possibilities set 10 pts

$$Y = \{(y_1, y_2) : y_2 \le e^{-y_1} - 1 \wedge y_1 \le 0\} ,$$

where $e^{\cdot}$ denotes the exponential function and $\wedge$ denotes the logical operator "and."

(a) Draw the graph of Y. *3 pts*



(b) Show whether or not Y is closed, convex, and satisfies free disposal. *3 pts*

Y is closed ½ since it contains all of its boundary points—i.e., bdy $Y \subset Y$. ½

Y is not convex ½ since, for example, $t(-1, e-1) + (1-t)(0,0)$ is not in Y for any $t \in (0,1)$. ½

Y satisfies free disposal ½ since, for any $y \in Y$, every y' such that $y' \le y$ is also in Y. ½

(c) Characterize the set of efficient points of Y. *1 pt*

$\{(y_1, y_2) : y_2 = e^{-y_1} - 1 \wedge y_1 \le 0\} =: E$ ½ ½

(d) What returns to scale property does Y exhibit, if any? *2 pts*

Non-decreasing returns to scale ½ but not non-increasing returns to scale ½ since any $y \in Y$ can be scaled up ½ (and remain feasible) but not necessarily down ½—e.g., $t(-1, e-1)$ is in Y for any $t > 1$ but is not in Y for any $t \in (0,1)$.

(e) If $(p_1, p_2) = (e, 1)$, which production plan will a profit-maximizing firm select from Y and how much profit will it make? *1 pt*

There is no profit-maximizing production plan $^{1/2}$ since for every $(y_1, y_2) \in E$, $(e,1)\cdot(y_1', y_2') > (e,1)\cdot(y_1,y_2)$ for any $(y_1', y_2') \in E$ such that $y_1' < y_1$ $^{1/2}$ (and $y_2' > y_2$).

2. Consider the constrained minimization problem 10 pts

$$c(w, y) = \min_x \{wx : x \geq g(y)\} ,$$

where $g(\cdot)$ is an increasing, twice continuously differentiable function.

(a) Explain the meaning of $g(y)$ in economic terms. *2 pts*

$g(y)$ is the technologically efficient level of input $^{1/2}$ for producing output level y. $^{1/2}$

Since $g(\cdot)$ is strictly monotonic $^{1/2}$ (increasing), it is invertible and $g^{-1}(\cdot)$ is the associated production function. $^{1/2}$

(b) State the necessary and sufficient conditions for $x^* > 0$ to solve the above constrained minimization problem. *2 pts*

Since $g(\cdot)$ is increasing, $^{1/2}$ $c(w,y) = wg(y)$, $^{1/2}$ which means that $x^* = g(y) > 0$ $^{1/2}$ and $g'(y) > 0$. $^{1/2}$

Now consider the unconstrained maximization problem

$$\pi(p, w) = \max_y \{py - c(w, y)\} .$$

(c) In the simplest possible terms, state the necessary and sufficient conditions for $y^* > 0$ to solve the preceding unconstrained maximization problem. *2 pts*

$$p - c_y(w, y^*) \overset{1/2}{=} 0 \Leftrightarrow p \overset{1/2}{=} wg'(y^*)$$

$$-c_{yy}(w, y^*) \underset{1/2}{<} 0 \Leftrightarrow wg''(y^*) \underset{1/2}{>} 0$$

(d) Determine the signs of the derivatives $\left.\frac{dy^*}{dp}\right|_{dw=0}$ and $\left.\frac{dy^*}{dw}\right|_{dp=0}$. *1.5 pts*

Totally differentiating the part-(c) first-order condition yields

$$dp \overset{1/2}{=} g'(y^*)dw + wg''(y^*)dy^*$$

$$\Rightarrow \left.\frac{dy^*}{dp}\right|_{dw=0} = \frac{1}{wg''(y^*)} \underset{1/2}{>} 0 \text{ and } \left.\frac{dy^*}{dw}\right|_{dp=0} = -\frac{g'(y^*)}{wg''(y^*)} \underset{1/2}{<} 0.$$

(e) Show that $\pi_p(p, w) = y^*$ and $\pi_w(p, w) = -x^*$. *2.5 pts*

By the definitions of π and y^*, $\pi(p,w) \equiv py^* - wg(y^*)$. $\frac{1}{2}$

Differentiating this expression with respect to p and w yields, respectively,

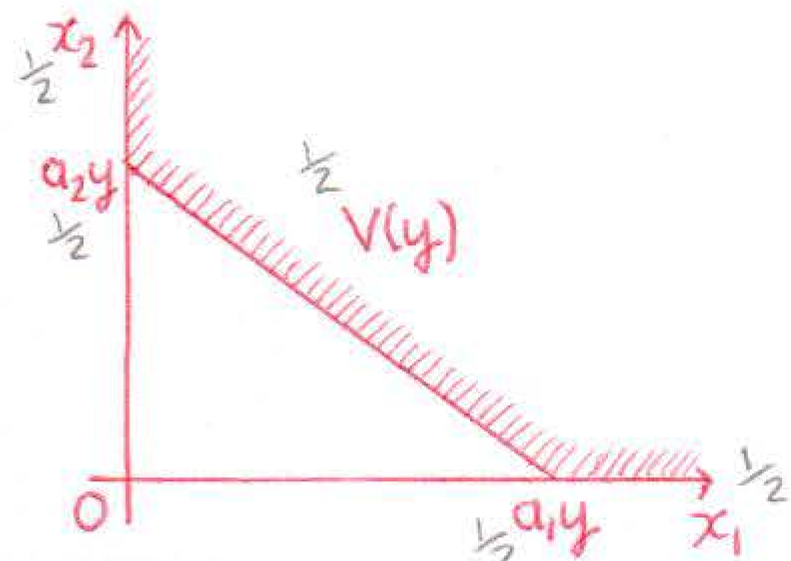
$\pi_p(p,w) = y^* + [p - wg'(y^*)]\frac{\partial y^*}{\partial p} = y^*$; $\frac{1}{2}$ $\frac{1}{2}$ by the part-(c) FOC, and

$\pi_w(p,w) = [p - wg'(y^*)]\frac{\partial y^*}{\partial w} - g(y^*) = -x^*$; $\frac{1}{2}$ $\frac{1}{2}$ by the FOCs of parts (c) & (b).

3. Consider the production function given by 10 pts

$$y = \frac{x_1}{a_1} + \frac{x_2}{a_2} \;, a_1 > 0, a_2 > 0.$$

(a) For y fixed, draw the graph of the input requirement set. *2.5 pts*

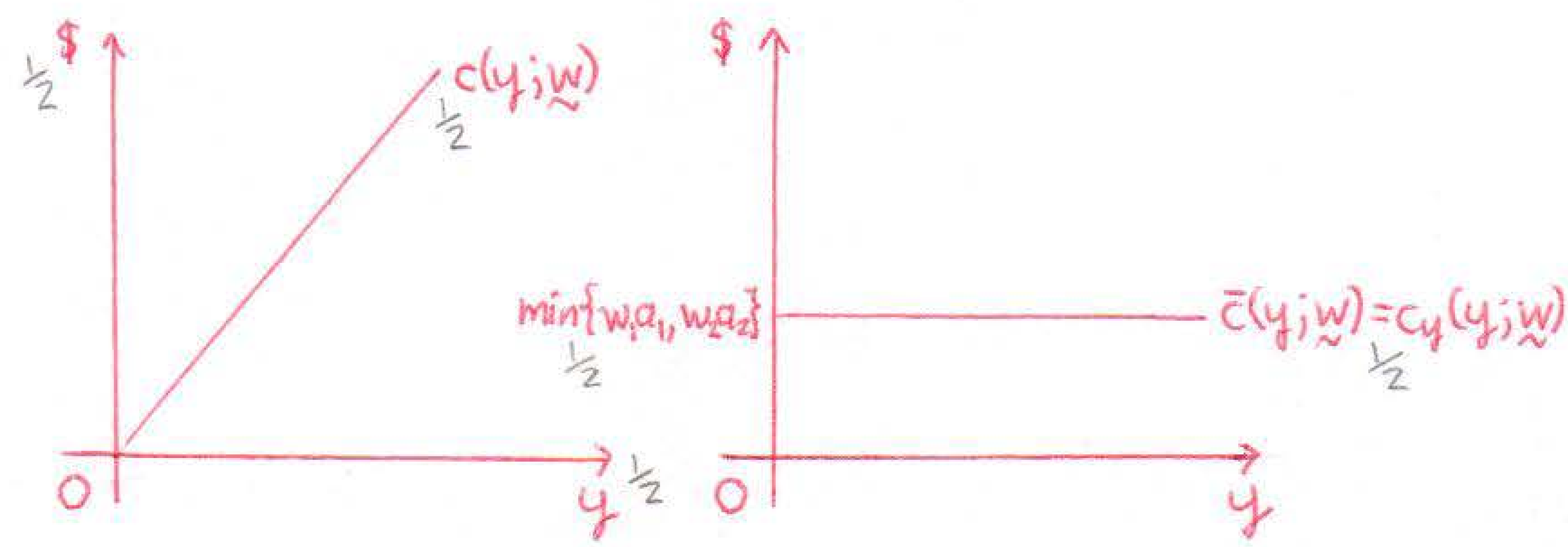


(b) Derive the associated cost function. *2 pts*

If $w_1a_1 < w_2a_2$, then $\underset{\sim}{x}^* = (a_1y, 0)$; $\frac{1}{2}$ if $w_1a_1 > w_2a_2$, then $\underset{\sim}{x}^* = (0, a_2y)$; $\frac{1}{2}$
if $w_1a_1 = w_2a_2$, then $\underset{\sim}{x}^*$ is an element of the straight line connecting $(a_1y, 0)$ and $(0, a_2y)$. $\frac{1}{2}$

Therefore, $c(w_1, w_2, y) = \min\{w_1a_1, w_2a_2\}y$. $\frac{1}{2}$

(c) For w_1 and w_2 fixed, draw the graphs of total cost, average total cost and marginal cost. *2.5 pts*



(d) State and prove the monotonicity, homogeneity, and curvature properties of the part-(b) cost function. *3 pts*

Since $\min\{w_1' a_1, w_2' a_2\} \geq \min\{w_1 a_1, w_2 a_2\}$ for any $\underset{\sim}{w}' \geq \underset{\sim}{w}$, $\frac{1}{2}$
$c(\underset{\sim}{w}, y)$ is non-decreasing in $\underset{\sim}{w}$. $\frac{1}{2}$

Since $\min\{t w_1 a_1, t w_2 a_2\} = t \min\{w_1 a_1, w_2 a_2\}$, $\frac{1}{2}$ $c(\underset{\sim}{w}, y)$ is homogeneous of degree one in $\underset{\sim}{w}$. $\frac{1}{2}$

Since $\min\{[t w_1 + (1-t) w_1'] a_1, [t w_2 + (1-t) w_2'] a_2\}$
$\geq \min\{t w_1 a_1, t w_2 a_2\} + \min\{(1-t) w_1' a_1, (1-t) w_2' a_2\}$, $\frac{1}{2}$
$c(\underset{\sim}{w}, y)$ is concave in $\underset{\sim}{w}$. $\frac{1}{2}$

End of Examination