Concordia University March 23, 2007

Engineering Differential Equations
Section J
Exam II (B)

ANSWER KEY

(1) (10 points) Solve the equation

$$xy' + y = \frac{1}{y^2}, \quad x > 0.$$

Solution: $xy' + y = 1/y^2$ is a Bernoulli equation with $n = -2$. Denoting $v = y^3$ and evaluating $v' = 3y^2y'$, the original equation becomes $v' + \frac{3}{x}v = \frac{3}{x}$. This is a linear first-order differential equation which has an integrating factor of x^3. So $x^3 v = x^3 + C$, where C is a constant, and thus $y(x) = \sqrt[3]{1 + \frac{C}{x^3}}$, C = arbitrary constant, is the general solution of the given Bernoulli equation.

□

(2) (10 points) A tank contains 200 liters of fluid in which 30 grams of salt is dissolved. Pure water is then pumped into the tank at a rate of 4 L/min; the well-mixed solution is pumped out at a rate of 3 L/min.

Solution: (a) Let $A(t)$ denote the number of grams of salt existent in the tank at time t. Then $A'(t) = R_{in} - R_{out} = 0 - R_{out}$ and therefore it satisfies the differential equation

$$\frac{dA}{dt} = -\frac{A}{200+t} \cdot 3.$$

This is a linear, and a separable, differential equation. Using the resolution for separable equations, we obtain that its general solution is

$$\ln A(t) = -3 \ln |200 + t\,| + C \quad \Rightarrow \quad A(t) = K\,\frac{1}{|200+t\,|^3},$$

where C, respectively K, is an arbitrary constant.

Since $A(0) = 30$, we find that $K = 30 \cdot (200)^3$. Consequently

$$A(t) = \frac{3 \cdot 200^3}{(200+t)^3}, \quad \text{thus} \quad A(5) = 30 \cdot \left(\frac{200}{205}\right)^3 \approx 27.86\,\text{g}.$$

(b) If $V(t)$ denotes the amount of water in the tank at time t, then $V(t)$ satisfies the differential equation $V'(t) = 1$. Hence $V(t) = t + C$, where $C = V(0) = 200$. Consequently, $V(t) = t + 200$ liters and $V(t) = 300 \;\Rightarrow\; t = 100$ minutes.

(c) At the instant when the tank overflows, the number of grams of salt in the tank are

$$A(100) = 30 \cdot \left(\frac{200}{300}\right)^3 = 30 \cdot \left(\frac{2}{3}\right)^3 \approx\; 8.89 \text{ grams}.$$

□

(3) **Solution:**

(a) "The functions $y_1(x) = x$ and $y_2(x) = x^2$ cannot form a fundamental set of any second-order homogeneous equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$." is FALSE.

(b) "If y_p is a particular solution of the nonhomogeneous second-order differential equation with constant coefficients $y'' - 4y' + 3y = g(x)$, then $y(x) = e^x + y_p(x)$ is also a solution to this differential equation." is TRUE.

(c) The set of functions $f_1(x) = e,\ f_2(x) = e^x,\ f_3(x) = e^{x^2}$ is linearly independent on the interval $(-\infty, \infty)$. We evaluate the Wronskian of the three functions:

$$W(f_1, f_2, f_3)(x) = \det \begin{pmatrix} e & e^x & e^{x^2} \\ 0 & e^x & 2xe^{x^2} \\ 0 & e^x & 2e^{x^2} + 4x^2e^{x^2} \end{pmatrix} = e \det \begin{pmatrix} e^x & 2xe^{x^2} \\ e^x & 2e^{x^2} + 4x^2e^{x^2} \end{pmatrix} = 2e^{1+x+x^2}(2x^2 - x + 1) \neq 0,$$

for all real x's. Thus, by Theorem 3.3, the functions f_1, f_2, f_3 are linearly independent.

(d) To solve the initial value problem $y'' + y' = 0,\ y(0) = 1,\ y'(0) = -1$, consider first the auxiliary equation of the DE $y'' + y' = 0$. This is $m^2 + m = 0$ which has the complex conjugate roots: $m_1 = i$ and $m_2 = -i$. Therefore the general solution of the DE is $y_{gen}(x) = C_1 \cos(x) + C_2 \sin(x)$, where C_1, C_2 are arbitrary constants. We will now use the initial conditions to find C_1 and C_2. We have that

$$y(0) = C_1 = 1 \quad \text{and} \quad y'(0) = C_2 = -1,$$

so $C_1 = 1$ and $C_2 = -1$. The solution to the given IVP is then

$$y(x) = \cos(x) - \sin(x).$$

□

(4) (10 points) Find the general solution of the differential equation

$$y'' + 2y' = 2x.$$

Solution: Consider first the associated homogeneous equation $y'' + 2y' = 0$. The roots of its auxiliary equation $m^2 + 2m = 0$ are $m_1 = 0$ and $m_2 = -2$. Consequently,

$$y_c(x) = C_1 + C_2e^{-2x},$$

where C_1 and C_2 are arbitrary constants.

To complete the problem, we need to find now a particular solution to the given non-homogeneous DE. To do this, we will use the method of undetermined coefficients. Given the form of $g(x) = 2x$, and that of $y_c(x)$ which overlaps with the form of $g(x)$, we assume that $y_p(x) = x(Ax + b)$. Then

$$y_p'(x) = 2Ax + B, \quad y_p''(x) = 2A,$$

so

$$2A + 4Ax + 2B = 2x \quad \Rightarrow \quad A = 1/2,\ B = -1/2.$$

Therefore $y_p(x) = \dfrac{x}{2}\,(x - 1)$, and the general solution of the given DE is

$$y_{gen}(x) = C_1 + C_2e^{-2x} + \frac{x}{2}\,(x - 1),$$

where C_1 and C_2 are arbitrary constants.

□