Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 1322D FINAL EXAMINATION

Prof : Weixuan Li **Duration: 3 hours**

Your seat number ________________________

Last name ______________ First name ________.. Student number _________

Instructions:

1. The exam is closed book. Only non-programmable and non-graphing calculators (TI 30 or equivalent) are allowed.
2. The exam has 11 questions in 12 pages (including this cover page) with a total of 60 marks.
3. Write your solution in the space provided for each question. If you need extra space to write your solution, use the back of the pages. Please clearly indicate where your answer is whenever you do this.
4. Check your answer carefully to avoid arithmetic calculation mistakes.

585, av. King Edward
Ottawa (Ontario) K1N 6N5 Canada

585 King Edward Avenue
Ottawa, Ontario K1N 6N5 Canada

(613) 562-5864 • Téléc./Fax (613) 562-5776
Courriel/Email: uomath@science.uottawa.ca

1. (5 marks) (a) (3 marks) Find the value of the improper integral $\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx$.

(b) (2 marks) Use comparison test to show that the improper integral $\int_1^\infty \frac{x+1}{\sqrt[3]{x^5+x^2}}\,dx$ is divergent.

2. (6 marks) Consider the region R bounded by the graph of $y = x^2$ and the graph of $y = 4x - 3$.

(a) (1 mark) Sketch this region.

(b) (2 marks) Find the area of this region.

(c) (3 marks) Find the volume of the solid obtained by revolving this region about the y axis.

3. (5 marks) Consider the initial-value problem: $y' = \frac{y}{t^2}$, $y(1) = 1$.

(a) (2 marks) Use Euler's method with step size $h = 0.1$ to find an approximation of $y(1.2)$.

(b) (3 marks) Solve this equation analytically by separating the variables.

1. (6 marks) A reservoir of capacity 50000 m^3 initially contains pure water. Every minute, 60 m^3 water with salt of concentration 0.3 kg / m^3, and 40 m^3 water with salt of concentration 0.05 kg / m^3 is added to the reservoir, and 100 m^3 of well mixed water overflows out from the reservoir. Let $Q(t)$ be the quantity of salt in the reservoir.

(a) (2 marks) Find a differential equation satisfied by $Q(t)$.

(b) (1 mark) Without solving this differential equation, find $\lim_{t\to\infty} Q(t)$.

(c) (1 marks) Without solving this differential equation, sketch the graph of $Q(t)$.

(d) (2 marks) Solve this equation analytically.

5. (6 marks) Determine whether each of the following series is convergent or divergent. Justify your answer by appropriate test method:

(a) $\sum_{n=1}^{\infty} \frac{3}{n2^n}$; (b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(1+\cos^2 n)}$; (c) $\sum_{n=1}^{\infty} \left(e^{1/n^2} - 1\right)$.

6. (5 marks) A tank, which has the shape of the lower half of a sphere with radius 2 m, is filled with water ($\rho = 1000$ kg / m^3). Find the work (in Joules) needed to pump out the water to a point 1 meter above the tank. ($g = 9.8$ m / sec^2)

7. (6 marks) Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^n \frac{(x-2)^n}{3^n n}.$$

8. (5 marks) (a) (3 marks) Find the Maclaurin series of the function $y = \dfrac{x^2}{(1-x)^2}$. What is the interval of convergence of this series?

(b) (2 marks) Use the Maclaurin series of this function to find $y^{(7)}(0)$.

You may need the binomial series $(1+x)^k = \sum_{n=0}^{\infty} \dfrac{k(k-1)\dots(k-n+1)}{n!} x^n$, which converge in the interval $(-1, 1)$.

9. (5 marks) Consider the initial-value problem

$$P' = \frac{1}{800}P(80 - P),\ P(0) = 15.$$

(a) (2 marks) Without solving the equation, sketch the graph of the solution to this initial-value problem. Show the asymptote(s) and mark the inflection point(s) of the solution, if any.

(b) (3 marks) Solve this equation analytically.

10. (6 marks) Consider 2 variable function $z = x^2 - y^2 + xy + 4x - 3y - 5$.

(a) (2 marks) Find the gradient vector of this function at the point (2, 1, 5).

(b) (2 marks) Find the directional derivative of z at point (2, 1, 5) in the direction of the vector $\mathbf{v} = \sqrt{5}\,\mathbf{i} + 2\,\mathbf{j}$.

(c) (2 marks) If $x = \cos t$ and $y = \sin t$, find partial derivatives $\frac{\partial z}{\partial t}$ at $t = \pi / 3$ **by the chain rule.**

11. (5 marks) Consider 2-variable function $z = x^2 e^{-y/x}$.

(a) (2 mark) Find the partial derivatives z_x and z_y.

(b) (3 marks) Find the equation of the plane tangent to the graph of this function at the point with $x = 1$, $y = 0$.