THE UNIVERSITY OF WESTERN ONTARIO
Department of Applied Mathematics

Last Name:__________ First Name: __________ Student Number:__________

Applied Mathematics 2415 Midterm Examination

Wednesday, Dec.18, 2013 Time: 2:00pm - 5:00pm

There are 10 questions and each is worth 10%. Not all questions are meant to be equally difficult. Calculators should not be necessary but are permitted. Show all work on the exam sheets provided. Use the back of the sheets if necessary. Answers with no work shown will receive little or no points. The numbers after each equation are meant to enumerate the equations and do not indicate, in any way, the value of the question.

Note there is a list of useful equations on the back **two** pages of this exam.

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1. Find an explicit solution, $y(x)$, of the differential equation subject.

2.2 #7

$$\frac{dy}{dx} = e^{3x+2y} \tag{1}$$

$$\frac{dy}{dx} = e^{3x+2y}$$

$$\frac{dy}{dx} = e^{3x} e^{2y}$$

$$e^{-2y}\,dy = e^{3x}\,dx$$ seperable 3

$$\int e^{-2y}\,dy = \int e^{3x}\,dx$$

$$-\frac{1}{2} e^{-2y} = \frac{1}{3} e^{3x} + C$$ 2 gave 1 if they left out C

$C' = -2C$

$$e^{-2y} = -\frac{2}{3} e^{3x} + C'$$ 2

$$-2y = \ln\left(-\frac{2}{3} e^{3x} + C'\right)$$

$$y = -\frac{1}{2} \ln\left(-\frac{2}{3} e^{3x} + C'\right)$$ 3

gave 1 if they used ln, but failed to use it correctly. In the end I had to dole out marks here based on how bad they abused ln

2,3 #24

2. Find the general solution $y(x)$ of the differential equation for $-1 < x < 1$ by using an appropriate integration factor. Note that by partial fraction decomposition $\frac{2}{x^2-1} = \frac{1}{x-1} + \frac{-1}{x+1}$; which you may find useful.

$$(x^2-1)^2\frac{dy}{dx} + 2y = (x+1)^2 \qquad (2)$$

$$\frac{dy}{dx} + \frac{2}{(x^2-1)}y = \frac{(x+1)^2}{(x^2-1)} =$$

1st order linear 1

$$P(x) = \frac{2}{(x^2-1)} \quad 1$$

$$Q(x) = \frac{(x+1)^2}{(x^2-1)^2} = \frac{(x+1)^2}{(x+1)(x-1)} = \frac{(x+1)}{(x-1)} \quad 1$$

$$\rho = e^{\int \frac{2}{x^2-1}dx} = e^{\int \frac{1}{x-1}dx + \int \frac{-1}{x+1}dx}$$

$$= e^{\ln(x-1) - \ln(x+1)}$$

$$= e^{\ln\left(\frac{x-1}{x+1}\right)}$$

$$= \frac{x-1}{x+1} \quad 3$$

$$\therefore \frac{d}{dx}\left(\frac{x-1}{x+1}y\right) = \left(\frac{x+1}{x-1}\right)\left(\frac{x-1}{x+1}\right) = 1 \quad 1$$

$$\frac{x-1}{x+1}y = x + C \quad 1$$

$$(x-1)y = x(x+1) + C(x+1)$$

$$\boxed{y = \frac{x(x+1)}{x-1} + \frac{C(x+1)}{(x-1)}}$$

or

$$\boxed{y = (x+C)\left(\frac{x+1}{x-1}\right)}$$

either is fine 2

2.4 # 23

3. Show that the given differential equation is exact (it is!) and then use that fact to determine the solution of the initial value problem.

$$(4y + 2t - 5)dt + (6y + 4t - 1)dy = 0 \quad y(-1) = 2 \qquad (3)$$

Show exact

$$\frac{d}{dy}(4y + 2t - 5) \overset{?}{=} \frac{d}{dt}(6y + 4t - 1) \quad 2$$

$$4 \overset{?}{=} 4 \quad \checkmark$$

$\therefore$ Soln is $f = c$

&

$$① \quad \frac{\partial f}{\partial t} = 4y + 2t - 5 \quad 1$$

$$② \quad \frac{\partial f}{\partial y} = 6y + 4t - 1 \quad 1$$

4 pts if they got these Backwards

-1 if forget soln is $\boxed{f(t,y) = C}$ not just $f(t,y)$

$$① \quad \frac{\partial f}{\partial t} = 4y + 2t - 5$$

$$f = 4yt + t^2 - 5t + h(y) \quad 1$$

Sub ① to ②

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(4yt + t^2 - 5t + h(y)\right) = 6y + 4t - 1 \quad 1$$

$$= \cancel{4t} + h'(y) = 6y + \cancel{4t} - 1$$

$$h'(y) = 6y - 1$$

$$h(y) = 3y^2 - y + D \quad 1$$

$$\therefore f = 4yt + t^2 - 5t + 3y^2 - y + D$$

Gen Soln is $\boxed{4yt + t^2 - 5t + 3y^2 - y = C}$ 1

I.C. at $t = -1$, $y = 2$ $\therefore C = 8$ $\boxed{4yt + t^2 - 5t + 3y^2 - y = 8}$ 2

3.3 #31

But harder +5 not -5

4. Find the characteristic equation and then give the general solution for the differential equation

$$y'' - 4y' + 5y = 0 \qquad (4)$$

$r^2 - 4r + 5 = 0$ 3

$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{4 \pm \sqrt{16 - 4\cdot(1)(5)}}{2}$

$= 2 \pm \frac{1}{2}\sqrt{16 - 20}$

$= 2 \pm \frac{1}{2}\sqrt{-4}$

$= 2 \pm i$ 2

$y_1 = e^{2x}\cos(x)$ 2

$y_2 = e^{2x} s\cdot(x)$

$y(x) = Ae^{2x}\cos(x) + Be^{2x} s(x)$ 3

If they found wrong roots but the idea of the solution after this is right, I give them 5 points (if they have real roots)

3.4 #8

5. Using the method of undetermined coefficients find the general solution of the differential equation.

$$4y'' - 4y' - 3y = \cos(2x) \qquad (5)$$

$\underline{y_H}$

$4r^2 - 4r - 3 = 0$

$r = \frac{3}{2}$

$r = -\frac{1}{2}$

$y_H = \cancel{A}\,c_1 e^{3/2 x} + \cancel{B}\,c_2 e^{-1/2 x}$ 3

$\underline{y_p}$ by meth u c.

$y_p^T = A\cos(2x) + B\sin(2x)$ 3

$\boxed{\text{not soln to A.H.ODE so ok}}$ 1

So $4(-4A\cos(2x) - B4\sin(2x)) - 4(-A2\sin(2x) + 2B\cos(2x))$

$-3(A\cos(2x) + B\sin(2x))$

~~cos(2x)~~ | $\cancel{-19A + 8B}$ $\cancel{8A - 19B = 0}$

$-16A\cos(2x) - 8B\cos(2x) - 3A\cos(2x) +$

$-16B\sin(2x) + 8A\sin(2x) - 3B\sin(2x) = \cos(2x)$

$\cos(2x) \Rightarrow \quad -19A - 8B = 1$ 2

$-19B + 8A = 0$

$\therefore A = \frac{-19}{425} \approx 0.045$

1 $B = \frac{-8}{425} \approx 0.019$

$$\boxed{\text{So } y(x) = C_1 e^{3/2 x} + C_2 e^{-1/2 x} + -\frac{19}{425}\cos(2x) - \frac{8}{425}\sin(2x)}$$ 1

3.5 # 12

6. Using variation of parameters solve the differential equation to find $y(x)$.

$$y'' - 2y' + y = \frac{e^x}{1+x^2} \qquad (6)$$

−1 if they add extra consts. to y_p.

AH ODE

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$ 1

$r = 1$, $r = 1$ $\quad y_1 = e^x$, $y_2 = xe^x$ 2

$y_p = u_1 y_1(x) + u_2 y_2(x)$

$u_1' = -\frac{y_2 f(x)}{W} \qquad u_2' = \frac{y_1 f(x)}{W}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \quad 1$$

$\therefore u_1' = -\frac{x\cancel{e^x}\cdot\cancel{e^x}}{\cancel{e^{2x}}(1+x^2)} = -\frac{x}{1+x^2}$ 1 $\qquad u_2' = \frac{\cancel{e^x}\cdot\cancel{e^x}}{\cancel{e^{2x}}\,1+x^2} = \frac{1}{1+x^2}$ 1

$\therefore u_1 = \int \frac{-x}{1+x^2}\,dx$ $\qquad u_2 = \int \frac{1}{1+x^2}\,dx$

let $u = 1+x^2$, $du = 2x\,dx$ $\qquad = \tan^{-1}(x)$ from table 1

$= -\int \frac{1}{2}\frac{1}{u}\,du$

(note const don't matter)

$= -\frac{1}{2}\ln(u)$

$= -\frac{1}{2}\ln(1+x^2)$ 1

$\therefore y_p = u_1 y_1 + u_2 y_2 = -\frac{1}{2}\ln(1+x^2)e^x + \tan^{-1}(x)xe^x$

$\therefore y(x) = C_1 e^x + C_2 xe^x + -\frac{1}{2}\ln(1+x^2)e^x + \tan^{-1}(x)xe^x$ 1 1

4.2 #31

7. Use Laplace transforms to solve the initial value problem

$$y' - y = 1 \quad (7)$$
$$y(0) = 0 \quad (8)$$

$$\mathcal{L}[y'] - \mathcal{L}[y] = \mathcal{L}[1]$$

$$s\mathcal{L}[y] - y(0) - \mathcal{L}[y] = \frac{1}{s} \quad 2$$

$$sY(s) - 0 - Y(s) = \frac{1}{s}$$

$$Y(s)(s-1) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s(s-1)} \quad 2$$

$$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}$$

$$1 = A(s-1) + Bs$$

$s = 0$

$$1 = A(-1) + 0$$

$$A = -1 \quad 1$$

$s = 1$

$$1 = A(0) + B$$

$$B = 1 \quad 1$$

$$Y(s) = \frac{-1}{s} + \frac{1}{s-1}$$

$$\boxed{y(t) = -1 + e^{t}}$$

2 2

4.3 #63

8. Use Laplace transforms to solve the differential equation

$$y' + y = f(t) \tag{9}$$
$$y(0) = 0 \tag{10}$$

where

$$f(t) = \begin{cases} 0, & 0 \le t < 1, \\ 5, & t \ge 1 \end{cases} \tag{11}$$

$f(t) = 5u(t-1)$ 1

$\mathcal{L}[y'] + \mathcal{L}[y] = \mathcal{L}[f(t)]$

$s\mathcal{L}[y] - \cancel{y(0)}^{\,0} + \mathcal{L}[y] = \mathcal{L}[5u(t-1)]$ 2

$(s+1)\mathcal{L}[y] = \dfrac{5e^{-s}}{s}$

$\mathcal{L}[y] = \dfrac{5e^{-s}}{s(s+1)}$ 3

PFD

$\dfrac{1}{s(s+1)} = \dfrac{A}{s} + \dfrac{B}{s+1}$

$1 = A(s+1) + Bs$

$A = 1$ 2

$B = -1$

$\therefore \mathcal{L}[y] = 5e^{-s}\left(\dfrac{1}{s} - \dfrac{1}{s+1}\right)$

$y(t) = \mathcal{L}^{-1}\left[5e^{-s}\left(\dfrac{1}{s} - \dfrac{1}{s+1}\right)\right]$

$= 5\mathcal{L}^{-1}\left[\dfrac{e^{-s}}{s}\right] - 5\mathcal{L}^{-1}\left[\dfrac{e^{-s}}{s+1}\right]$

$= 5u(t-1) - 5e^{-(t-1)}u(t-1)$

1 1

4.5 #6 modified by typo

9. Use Laplace transforms to solve the differential equation

$$\begin{aligned} y''(t) + y'(t) &= \delta(t-2\pi) + \delta(t-4\pi) && (12)\\ y(0) &= 1 && (13)\\ y'(0) &= 0 && (14) \end{aligned}$$

$$\mathcal{L}[y''] + \mathcal{L}[y'] = \mathcal{L}[\delta(t-2\pi)] + \mathcal{L}[\delta(t-4\pi)]$$

$$s^2\mathcal{L}[y] - \cancel{y'(0)}^{0} - s\,\cancel{y(0)}^{1} + s\mathcal{L}[y] - \cancel{y(0)}^{1} = e^{-2\pi s} + e^{-4\pi s}$$ 1 2

$$s^2\mathcal{L}[y] - s + s\mathcal{L}[y] - 1 = e^{-2\pi s} + e^{-4\pi s}$$

$$(s^2 + s)\mathcal{L}[y] - (s+1) = e^{-2\pi s} + e^{-4\pi s}$$

$$s(s+1)\mathcal{L}[y] = (s+1) + e^{-2\pi s} + e^{-4\pi s}$$

$$\mathcal{L}[y] = \frac{1}{s} + \frac{1}{s(s+1)}\left(e^{-2\pi s} + e^{-4\pi s}\right)$$ 3

PFD

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

As #8

$$= \frac{1}{s} + \frac{-1}{s+1}$$

1

$$\mathcal{L}[y] = \frac{1}{s} + \frac{1}{s}\left(e^{-2\pi s} + e^{-4\pi s}\right) - \frac{1}{s+1}\left(e^{2\pi s} + e^{-4\pi s}\right)$$

$$= 1 + u(t-2\pi) + -e^{-(t-2\pi)}u(t-2\pi)$$ 1

$$u(t-4\pi) - e^{-(t-4\pi)}u(t-4\pi)$$ 1 1

$$= 1 + \left(1 - e^{-(t-2\pi)}\right)u(t-2\pi) + \left(1 - e^{-(t-4\pi)}\right)u(t-4\pi)$$

4.6 #3

me

10. **Use Laplace transforms** to solve the system of differential equations.

$$
\begin{aligned}
① \quad x'(t) &= x(t) - 2y(t) && (15)\\
② \quad y'(t) &= 5x(t) - y(t) && (16)\\
x(0) &= -1 && (17)\\
y(0) &= 2 && (18)
\end{aligned}
$$

① $\mathcal{L}[x'] = \mathcal{L}[x] - 2\mathcal{L}[y]$

$s X(s) - x(0) = X(s) - 2Y(s)$

$s X(s) + 1 = X(s) - 2Y(s)$

$(s-1)X(s) + 2Y(s) = -1$

② $\mathcal{L}[y'] = 5\mathcal{L}[x] - \mathcal{L}[y]$

$s\mathcal{L}[y] - y(0) = 5\mathcal{L}[x] - \mathcal{L}[y]$

$sY(s) - 2 = 5X(s) - Y(s)$

$-5X(s) + (s+1)Y(s) = 2$

① $(s-1)X(s) + 2Y(s) = -1$ 2 eq

② $-5X(s) + (s+1)Y(s) = 2$ 2 unknown

② $X(s) = \frac{2}{5} - \frac{(s+1)Y(s)}{5}$

$(s-1)\left[\frac{2}{5} - \frac{(s+1)Y(s)}{5}\right] + 2Y(s) = -1$

$\frac{2}{5}(s-1) - \frac{(s^2-1)}{5}Y(s) + 2Y(s) = -1$

$Y(s)\left[\frac{s^2-1}{5} + 2\right] = -1 + \frac{2}{5}(s-1)$

$Y(s)\left[\frac{s^2-1+10}{5}\right] = +\frac{2}{5}s - \frac{7}{5}$

#2

10. **Use Laplace transforms** to solve the system of differential equations.

$$x'(t) = x(t) - 2y(t) \quad (15)$$
$$y'(t) = 5x(t) - y(t) \quad (16)$$
$$x(0) = -1 \quad (17)$$
$$y(0) = 2 \quad (18)$$

$$y(s)(s^2+9) = -2s - 7$$

$$y(s) = \frac{-2s-7}{(s^2+9)} = \frac{-2s}{s^2+9} - \frac{7}{s^2+9}$$

$$\boxed{y(t) = 2\cos(3t) - \frac{7}{3}\sin(3t)}$$

from $y'(t) = 5\cancel{\,}x(t) - y(t)$

$$x(t) = \frac{1}{5}\left(y'(t) + y(t)\right)$$

$$= \frac{1}{5}\left(-6\sin(3t) - 7\cos(3t) + 2\cos(3t) - \frac{7}{3}\sin(3t)\right)$$

$$\boxed{x(t) = -\cos(3t) - \frac{5}{3}\sin(3t)}$$