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Quiz is out of **17 marks**. Duration is **50 minutes**. Only **non-programmable** calculators are permitted. Books and/or notes are not permitted.

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1. [5 marks] Find the general solution to

$$3x^2 - 2xy + 2 + (6y^2 - x^2 + 3)\frac{dy}{dx} = 0.$$

**Solution:**  $P(x, y) = 3x^2 - 2xy + 2$  and  $Q(x, y) = 6y^2 - x^2 + 3$  so that  $P_y(x, y) = -2x = Q_x(x, y)$  and the equation is exact [1]. Hence, there exists a function  $f = f(x, y)$  satisfying  $f_x(x, y) = P(x, y) = 3x^2 - 2xy + 2$  and  $f_y(x, y) = Q(x, y) = 6y^2 - x^2 + 3$  [1]. Integrating with respect to  $x$ , for instance, gives

$$f(x, y) = \int f_x(x, y) dx = \int 3x^2 - 2xy + 2 dx = x^3 - x^2y + 2x + g(y) [1].$$

Differentiating with respect to  $y$  then yields

$$f_y(x, y) = -x^2 + g'(y) = 6y^2 - x^2 + 3 \Rightarrow g'(y) = 6y^2 + 3 [1].$$

Thus,  $g(y) = 2y^3 + 3y + C$ , and the general solution is

$$f(x, y) = x^3 - x^2y + 2x + 2y^3 + 3y + C = 0 [1].$$

2. [5 marks] Find the general solution to

$$\frac{dy}{dx} = \frac{x+y}{x-y}.$$

**Solution:** Dividing the numerator and denominator by  $x$ , we see that

$$\frac{dy}{dx} = \frac{1+y/x}{1-y/x}$$

so the equation is homogenous [1]. Substituting  $u = y/x$  gives  $y' = u + xu'$  and the equation becomes

$$u + xu' = \frac{1+u}{1-u} \Rightarrow xu' = \frac{1+u^2}{1-u} \Rightarrow \frac{1-u}{1+u^2}u' = \frac{1}{x} \text{ [1].}$$

Integrating with respect to  $x$  we obtain

$$\begin{aligned} \ln|x| + C &= \int \frac{1-u}{1+u^2} dx \\ &= \int \frac{1}{1+u^2} dx - \int \frac{u}{1+u^2} dx \\ &= \arctan(u) - \frac{1}{2} \ln(u^2 + 1) \text{ [2].} \end{aligned}$$

Hence, the general solution is

$$\ln|x| + C = \arctan\left(\frac{y}{x}\right) - \frac{1}{2} \ln\left(\left(\frac{y}{x}\right)^2 + 1\right) \text{ [1].}$$

3. [7 marks] Solve the initial value problem

$$\frac{dy}{dx} = ay - by^3, \quad y(0) = 1,$$

where  $a, b > 0$  are positive constants.

**Solution:** The equation is Bernoulli with  $\alpha = 3$ , so we make the substitution  $u = y^{1-\alpha} = y^{-2}$  [1]. Then  $y = u^{-1/2}$  and

$$y' = -\frac{1}{2}u^{-3/2}u' = au^{-1/2} - bu^{-3/2} \Rightarrow u' + 2au = 2b \text{ [1].}$$

This equation is linear with integrating factor

$$I(x) = e^{\int P(x)dx} = e^{\int 2adx} = e^{2ax} \text{ [1].}$$

Hence, the solution is

$$u(x) = \frac{1}{e^{2ax}} \left[ \int 2be^{2ax} dx \right] = \frac{1}{e^{2ax}} \left[ \frac{b}{a}e^{2ax} + C \right] = \frac{b}{a} + Ce^{-2ax} \text{ [1].}$$

Solving for  $y$ , we obtain

$$y(x) = \pm \frac{1}{\sqrt{b/a + Ce^{-2ax}}} \text{ [1].}$$

Plugging in our initial data,

$$y(0) = 1 = \pm \frac{1}{\sqrt{b/a + C}} \Rightarrow C = 1 - \frac{b}{a} \text{ [0.5]}$$

and we take the positive square root [0.5]. The solution is therefore

$$y(x) = \frac{1}{\sqrt{e^{-2ax} + b/a(1 - e^{-2ax})}} \text{ [1].}$$