Midterm 1 Exam Solutions

1. If $P(A) = \frac{1}{3}$ and $P(B^c) = \frac{1}{4}$. Can A, B be mutually exclusive? Explain.

There were a number of ways of answering this question. We'll consider one of those ways here.
If A and B are mutually exclusive then $AB = \emptyset$ and $P(AB) = 0$. By the general addition rule, $P(A \cup B) = P(A) + P(B) - P(AB)$. We know that as a consequence of the rules of probability, P(any event) ≤ 1. This implies that $P(A \cup B) \leq 1 \iff P(A) + P(B) - P(AB) \leq 1$ $(*)$.

We are told $P(A) = \frac{1}{3}$ and $P(B^c) = \frac{1}{4} \implies P(B) = 1 - P(B^c) = 1 - \frac{1}{4} = \frac{3}{4}$.
So considering $(*)$ again,

$$\frac{1}{3} + \frac{3}{4} - P(AB) \leq 1$$
$$P(AB) \geq \frac{1}{12} \neq 0 \quad \text{(if disjoint then } = 0)$$

$\therefore A + B$ cannot be mutually exclusive.

[Note: Mutually exclusive events $\not\equiv$ Independent events
Many people have these two concepts confused and so were using incorrect arguments in this question.
Mutually exclusive $\implies AB = \emptyset$ or $P(AB) = 0$ i.e. no element in common.
Independent $\implies P(AB) = P(A)P(B)$ (which is only $= 0$ if either $P(A) = 0$ or $P(B) = 0$)]

2. The letters of the word STATISTICS are arranged in a random order. How many distinct arrangements are there? What is the probability that an S occurs at each end?

The number of arrangements of 10 distinct letters is $P_{10}^{10} = 10!$. However we don't have 10 distinct letters: there are 3 S's, 3 T's, and 2 I's.

$$\begin{aligned}
\therefore \text{ Number of distinct arrangements} &= \frac{10!}{3!3!2!} = 50,400 \\
P(\text{S occurring at each end}) &= \\
&\frac{\binom{\text{Number of distinct arrangements}}{\text{with S occurring at each end}}}{\text{Number of distinct arrangements}} \\
&= \frac{\frac{8!}{3!2!}}{\frac{10!}{3!3!2!}} \\
&= \frac{8!3!}{10!} \\
&= \frac{1}{15} \text{ or } 0.0667
\end{aligned}$$

3. The flash mechanism on camera A fails on 10% of shots, while that on camera B fails on 5% of shots. The two cameras being identical in appearance, a photographer selects one at random and takes 10 indoor shots using the flash.
a) Give the probability that the flash machine fails exactly twice. What assumption(s) are you making?
b) Given that the flash mechanism failed exactly twice, what is the probability camera A was selected?

a) Let F = number of times the camera flash fails,
A = {camera A selected}, B = {camera B selected}
We are asked to calculate $P(F = 2)$.
The assumptions here are:
1. Photographer selects camera randomly
$\Longrightarrow P(A) = P(B) = 1/2$
2. Shots are independent
3. Probability flash fails is the same for every shot.
If (2) and (3) assumed, then $F \sim Binomial(n, p)$ where $n = 10$ and $p =$ probability of failed shot, which depends on which camera was selected. i.e. $p_A = 0.1$ and $p_B = 0.05$
So

$$\begin{aligned}
P(F = 2) &= P(F = 2|A)P(A) + P(F = 2|B)P(B) \\
&= \binom{10}{2} p_A^2 (1 - p_A)^8 \times \frac{1}{2} + \binom{10}{2} p_B^2 (1 - p_B)^8 \times \frac{1}{2} \\
&= 45 \times (0.1)^2 (0.9)^8 \times \frac{1}{2} + 45 \times (0.05)^2 (0.95)^8 \times \frac{1}{2} \\
&= 0.1342
\end{aligned}$$

b) Let A = {camera A selected} and F = number of times the camera flash fails

Then we are asked to calculate

$$
\begin{aligned}
P(A|F=2) &= \frac{P(A\ and\ F=2)}{P(F=2)} \\
&= \frac{P(F=2|A)P(A)}{P(F=2)} \\
&= \frac{\binom{10}{2}(0.1)^2(0.9)^8 \times \frac{1}{2}}{0.1342_{\leftarrow part\ a}} \\
&= 0.7219
\end{aligned}
$$

4. The length of time (in minutes) it takes to wait for the bus has the following probability density function:
$f(x) = ke^{-kx}$ for $x \geq 0$
If the probability that you have to wait more than 10 minutes is $\frac{1}{e}$, what is the expected waiting time?

$$P(X > 10) = \int_{10}^{\infty} ke^{-kx}dx$$

$$\text{Integrate by substitution, let } u = kx \implies \frac{1}{k}du = dx$$

$$
\begin{aligned}
\text{so } \int_{u=10k}^{\infty} ke^{-u}\frac{1}{k}du &= \int_{u=10k}^{\infty} e^{-u}du \\
&= -e^{-u}\Big|_{10k}^{\infty} = e^{-10k}
\end{aligned}
$$

$$P(X > 10) = \frac{1}{e} \quad \text{so } e^{-10k} = \frac{1}{e} \implies k = \frac{1}{10}$$

$$
\begin{aligned}
E(X) &= \int_0^{\infty} kxe^{-kx}dx \\
&= \frac{1}{10}\int_0^{\infty} xe^{-\frac{x}{10}}dx
\end{aligned}
$$

Integrate by parts, let $u = x$

$$
\begin{aligned}
du &= dx \\
\\
dv &= e^{-\frac{x}{10}} dx \\
v &= -10e^{-\frac{x}{10}} \\
E(X) &= -\frac{1}{10}(10xe^{-\frac{x}{10}})\Big|_0^\infty + \int 10e^{-\frac{x}{10}} dx \\
&= -\frac{1}{10}(10xe^{-\frac{x}{10}} + 100e^{-\frac{x}{10}})\Big|_0^\infty \\
&= (xe^{-x/10} - 10e^{-x/10})\Big|_0^\infty \\
&= 10e^{-0/10} \\
&= 10
\end{aligned}
$$

5. A fair die is rolled twice and the scores are noted. Let X equal the first score minus the second score. Determine the probability function of X, and hence, determine $P(X > 2)$.

X = {first side - second side }
The sample space for X is {-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5}. The probability function for X is

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
p(x)	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36

e.g. $p(-3) = \frac{\{(1,4)\cap(2,5)\cap(3,6)\}}{36} = \frac{3}{36} = \frac{1}{12}$ (36 outcomes altogether $= 6 \times 6$)

$$
\begin{aligned}
P(X > 2) &= P(X = 3) + P(X = 4) + P(X = 5) \\
&= p(3) + p(4) + p(5) \\
&= \frac{1}{12} + \frac{1}{18} + \frac{1}{36} \\
&= \frac{1}{6}
\end{aligned}
$$

6. The number of spectators that turn up to watch a local ice hockey team is known to have a Poisson distribution with mean 10. The admission charge is \$3 per spectator and the gate person is paid \$16 for collecting the admission money. What is
a) the expected profit?
b) the standard deviation of the profit?

c) the probability of making a loss?

a) Let X = number of spectators. We are told $X \sim Poisson(10) \implies E(X) = 10$ and $Var(X) = 10$.
Let P = profit. Then $P = 3X - 16$. (\$3 per spectator entry fee and \$16 paid to gate person.

$$\begin{aligned} \text{So } E(P) &= E(3X - 16) \\ &= 3E(X) - 16 \\ &= 3 \times 10 - 16 \\ &= \$14 \end{aligned}$$

b) Standard deviation of the profit

$$\begin{aligned} &= \sqrt{Var(P)} \\ &= \sqrt{Var(3X - 16)} \\ &= \sqrt{3^2 Var(X)} \\ &= 3\sqrt{Var(X)} \\ &= 3\sqrt{10} \\ &\approx \$9.49 \end{aligned}$$

[Note: $Var(aX) = a^2 Var(X)$ not $aVar(X)$. Also this question asked for standard deviation of profit not number of spectators.]

c)

$$\begin{aligned} P(\text{making a loss}) &= P(\text{Profit} < 0) \\ &= P(P < 0) \\ &= P(3X - 16 < 0) \\ &= P(X < 16/3) \\ &\text{But X needs to be an integer, so} \\ &= P(X \leq 5) \\ &= \sum_{x=0}^{5} P(X = x) \\ \text{X is } Poisson(10) \text{ so } P(X = x) &= \frac{e^{-10}10^x}{x!} \\ P(\text{making a loss}) &= \sum_{x=0}^{5} \frac{e^{-10}10^x}{x!} \\ &\approx 0.067 \end{aligned}$$

7) Let the random variable $N \sim Poisson(\lambda)$. And given that $N = n$, the random X follows a Binomial distribution: $X \sim Bin(N, p)$. What is the distribution of X? Note that the parameter N in the Binomial distribution is a random variable. Hint: Derive the probability mass function of X.
You may find this result useful:
$\sum_{j=0}^{\infty} \frac{x^j}{j!} = e^x$

Note that $P(X = k|N = n) = 0$ for $n < k$. Utilizing the law of total probability, we can write:

$$\begin{aligned} P(X = k) &= \sum_{n=0}^{\infty} P(X = k|N = n)P(N = n) \\ &= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \times \frac{\lambda^n e^{-\lambda}}{n!} \\ &= \frac{\lambda^k e^{-\lambda} p^k}{k!} \sum_{n=k}^{\infty} \frac{(\lambda(1-p))^{n-k}}{(n-k)!} \end{aligned}$$

Set $n - k = n^{'}$ and make a change of variables in the summation to get:

$$= \frac{\lambda^k e^{-\lambda} p^k}{k!} \sum_{n^{'}=0}^{\infty} \frac{(\lambda(1-p))^{n^{'}}}{n^{'}!}$$

Recalling that $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$, we can reduce this to:

$$\begin{aligned} &= \frac{(\lambda p)^k e^{-\lambda}}{k!} e^{\lambda(1-p)} \\ &= \frac{(\lambda p)^k e^{-\lambda p}}{k!} \end{aligned}$$

Which implies that $X \sim Poisson(\lambda p)$