Assignment 10

Due Thursday, Nov. 20, 2014

Note: there are 5 problems

1. Give a complete proof of the following statement.

The set of all rational numbers $\mathbb{Q}$ *is countable.*

2. Consider the following statement: If $a^2 + 9 \geq 6a$ then $a \geq 3$.

(i) Show, by a counterexample, that this statement is incorrect.

(ii) Explain what is wrong with the following "proof" of the above statement.

Backward: if $a \geq 3$ then $a - 3 \geq 0$, so $(a-3)^2 \geq 0$, $a^2 - 6a + 9 \geq 0$ and finally $a^2 + 9 \geq 6a$.

Forward: if $a^2 + 9 \geq 6a$ then $a^2 - 6a + 9 \geq 0$, hence $(a-3)^2 \geq 0$ and taking square roots we conclude $a - 3 \geq 0$ and so $a \geq 3$.

3. Suppose c and d are positive integers. Show that if the remainder of c divided by d is 1, the same is true for the remainder of c^2.

4. Suppose A, B, C are sets and consider the functions (or mappings): $f : A \longrightarrow B$ and $g : B \longrightarrow C$.

(i) Show that if f and g are both injective then so is their composition $g \circ f$

(ii) Show that if f and g are surjective then so is their composition $g \circ f$.

[Recall that $(g \circ f)(a) = g(f(a))$.]

5. Use the quantifiers $\forall$ and $\exists$ to rewrite the following statements:

(a) Let $f(x)$ be a continuous, real-valued function of the real variable x. If $f(a) = f(b) = 0$ for any two points $a, b \in \mathbb{R}$, such that $a < b$, then $f'(c) = 0$ for some point $c \in (a, b)$.

(Note that $f'(x)$ denotes the derivative of $f(x)$.)

(b) Between any two rational numbers r_1 and r_2, one can find at least one other rational number r.