Assignment 1 Solutions

1. For each of the following statements use a truth table to determine whether it is a tautology, a contradiction, or a contingency.

   (a) $\underbrace{(p \wedge \underbrace{(\neg(\neg p \vee q))}_{\mathbf{a}})}_{\mathbf{b}} \vee \underbrace{(p \wedge q)}_{\mathbf{c}}$

   Solution: Contingency.

p	q	$\neg p$	$\neg p \vee q$	$\overbrace{\neg(\neg p \vee q)}^{\mathbf{a}}$	$\overbrace{p \wedge \mathbf{a}}^{\mathbf{b}}$	$\overbrace{p \wedge q}^{\mathbf{c}}$	$\mathbf{b} \vee \mathbf{c}$
T	T	F	T	F	F	T	T
T	F	F	F	T	T	F	T
F	T	T	T	F	F	F	F
F	F	T	T	F	F	F	F

   (b) $(\underbrace{(p \to r)}_{\mathbf{a}} \vee \underbrace{(q \to r)}_{\mathbf{b}}) \to \underbrace{((p \vee q) \to r)}_{\mathbf{c}}$

   Solution: Contingency.

p	q	r	$\overbrace{p \to r}^{\mathbf{a}}$	$\overbrace{q \to r}^{\mathbf{b}}$	$\mathbf{a} \vee \mathbf{b}$	$p \vee q$	$\overbrace{(p \vee q) \to r}^{\mathbf{c}}$	$(\mathbf{a} \vee \mathbf{b}) \to \mathbf{c}$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F	T
T	F	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	F	F	T	T	T	F	F
F	T	F	T	F	T	T	F	F
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

2. For each of the following logical equivalences state whether it is valid or invalid. If invalid then give a counterexample (*e.g.*, based on a truth table). If valid then give an algebraic proof using logical equivalences from Tables 6, 7, and 8 from Section 1.3 of textbook.

   (a) $p \to (q \to r) \equiv q \to (\neg p \lor r)$

   Solution: Valid.

   $$\begin{array}{rll} & p \to (q \to r) & \\ \equiv & \neg p \lor (q \to r) & \text{law for conditional} \\ \equiv & \neg p \lor (\neg q \lor r) & \text{law for conditional} \\ \equiv & (\neg p \lor \neg q) \lor r & \text{associativity} \\ \equiv & (\neg q \lor \neg p) \lor r & \text{commutativity} \\ \equiv & \neg q \lor (\neg p \lor r) & \text{associativity} \\ \equiv & q \to (\neg p \lor r) & \text{law for conditional} \end{array}$$

   (b) $(p \to q) \lor (p \to r) \equiv ((p \lor q) \to r)$

   Solution: Invalid.
   If $p = T$, $q = T$, and $r = F$ then the LHS is True, while the RHS is False.

   (c) $((p \lor q) \land (\neg p \lor r)) \equiv (q \lor r)$

   Solution: Invalid.
   If $p = T$, $q = T$, and $r = F$ then the LHS is False, while the RHS is True.

3. For each of the statements below, write it down in the form "*if p then q*," then write down the converse statement, and finally the contrapositive.

Recall that the contrapositive is equivalent to the original implication, while in general the converse is not equivalent to the original implication.

(a) A positive integer is a prime only if it has no divisors other than 1 and itself.

Solution: If a positive integer is a prime, then it has no divisors other than 1 and itself.

Converse: If a positive integer has no divisors other than 1 and itself, then it is prime.

Contrapositive: If a positive integer has divisors other than 1 and itself, then it is not prime.

(b) To get an A in this class, it is necessary to do all the assignments.

Solution: If you want to get an A in this class, then you have to do all the assignments.

Converse: If you do all the assignments, then you will get an A in this class.

Contrapositive: If you don't do all the assignments, then you cannot get an A in this class.

(c) Being born in Canada is a sufficient condition for Canadian citizenship.

Solution: If you are born in Canada, then you can get Canadian citizenship.

Converse: If you can get Canadian citizenship, then you were born in Canada.

Contrapositive: If you cannot get Canadian citizenship, then you were not born in Canada.

(d) You will reach the summit unless you begin your climb too late.

Solution: If you do not begin your climb too late, then you will reach the summit.

If you did not reach the summit, then you began your climb too late.

Converse: If you reached the summit, then you did not begin your climb too late.

Contrapositive: If you did not reach the summit, then you began your climb too late.

4. A set of propositions is *consistent* if there is an assignment of truth values to each of the variables in the propositions that makes each proposition true. Is the following set of propositions consistent?

The system is in multi-user state if and only if it is operating normally.

$$MS \leftrightarrow ON$$

If the system is operating normally, the kernel is functioning.

$$ON \rightarrow KF$$

The kernel is not functioning or the system is in interrupt mode.

$$\neg KF \vee IM$$

If the system is not in multiuser state, then it is in interrupt mode.

$$\neg MS \rightarrow IM$$

The system is in interrupt mode.

$$IM$$

Solution: The set

$$\{MS \leftrightarrow ON,\ ON \rightarrow KF,\ \neg KF \vee IM,\ \neg MS \rightarrow IM,\ IM\}$$

is satisfiable. Proof:

MS	ON	KF	IM	$MS \leftrightarrow ON$	$ON \rightarrow KF$	$\neg KF \vee IM$	$\neg MS \rightarrow IM$
T	T	T	T	T	T	T	T

5. Let P and Q be predicates on the set S, where S has three elements, say, $S = \{a, b, c\}$. Then the statement $\forall x P(x)$ can also be written in full detail as $P(a) \wedge P(b) \wedge P(c)$. Rewrite each of the statements below in a similar fashion, using P, Q, and logical operators, but without using quantifiers.

(a) $\forall x, y(P(x) \vee Q(y)) \equiv \forall x\Big(\forall y\big(P(x) \vee Q(y)\big)\Big)$

Solution:

$$\begin{aligned}
& \forall x\Big(\forall y\big(P(x) \vee Q(y)\big)\Big) \\
\equiv\quad & \forall y\Big(P(a) \vee Q(y)\Big) \bigwedge \forall y\Big(P(b) \vee Q(y)\Big) \bigwedge \forall y\Big(P(c) \vee Q(y)\Big) \\
\equiv\quad & \Big[\Big(P(a) \vee Q(a)\Big) \wedge \Big(P(a) \vee Q(b)\Big) \wedge \Big(P(a) \vee Q(c)\Big)\Big] \\
\bigwedge\quad & \Big[\Big(P(b) \vee Q(a)\Big) \wedge \Big(P(b) \vee Q(b)\Big) \wedge \Big(P(b) \vee Q(c)\Big)\Big] \\
\bigwedge\quad & \Big[\Big(P(c) \vee Q(a)\Big) \wedge \Big(P(c) \vee Q(b)\Big) \wedge \Big(P(c) \vee Q(c)\Big)\Big] \\
\equiv\quad & \Big[P(a) \vee \Big(Q(a) \wedge Q(b) \wedge Q(c)\Big)\Big] \\
\bigwedge\quad & \Big[P(b) \vee \Big(Q(a) \wedge Q(b) \wedge Q(c)\Big)\Big] \\
\bigwedge\quad & \Big[P(c) \vee \Big(Q(a) \wedge Q(b) \wedge Q(c)\Big)\Big]
\end{aligned}$$

(b) $\exists x P(x) \wedge \exists x Q(x)$

Solution:

$$\begin{aligned}
& \exists x P(x) \wedge \exists x Q(x) \\
\equiv\quad & \Big(P(a) \vee P(b) \vee P(c)\Big) \wedge \Big(\exists x Q(x)\Big) \\
\equiv\quad & \Big(P(a) \vee P(b) \vee P(c)\Big) \wedge \Big(Q(a) \vee Q(b) \vee Q(c)\Big)
\end{aligned}$$

(c) $\exists x, y(P(x) \wedge Q(y)) \equiv \exists x\Big(\exists y\big(P(x) \wedge Q(y)\big)\Big)$

Solution:

$$
\begin{array}{ll}
 & \exists x\Big(\exists y\big(P(x) \wedge Q(y)\big)\Big) \\
\equiv & \exists y\Big(P(a) \wedge Q(y)\Big) \bigvee \exists y\Big(P(b) \wedge Q(y)\Big) \bigvee \exists y\Big(P(c) \wedge Q(y)\Big) \\
\equiv & \Big[\Big(P(a) \wedge Q(a)\Big) \vee \Big(P(a) \wedge Q(b)\Big) \vee \Big(P(a) \wedge Q(c)\Big)\Big] \\
\bigvee & \Big[\Big(P(b) \wedge Q(a)\Big) \vee \Big(P(b) \wedge Q(b)\Big) \vee \Big(P(b) \wedge Q(c)\Big)\Big] \\
\bigvee & \Big[\Big(P(c) \wedge Q(a)\Big) \vee \Big(P(c) \wedge Q(b)\Big) \vee \Big(P(c) \wedge Q(c)\Big)\Big] \\
\equiv & \Big[P(a) \wedge \Big(Q(a) \vee Q(b) \vee Q(c)\Big)\Big] \\
\bigvee & \Big[P(b) \wedge \Big(Q(a) \vee Q(b) \vee Q(c)\Big)\Big] \\
\bigvee & \Big[P(c) \wedge \Big(Q(a) \vee Q(b) \vee Q(c)\Big)\Big]
\end{array}
$$

(d) $\forall x \exists y(P(x) \wedge Q(y)) \equiv \forall x\Big(\exists y\big(P(x) \wedge Q(y)\big)\Big)$

Solution:

$$
\begin{array}{ll}
 & \forall x\Big(\exists y\big(P(x) \wedge Q(y)\big)\Big) \\
\equiv & \exists y\Big(P(a) \wedge Q(y)\Big) \bigvee \exists y\Big(P(b) \wedge Q(y)\Big) \bigvee \exists y\Big(P(c) \wedge Q(y)\Big) \\
\equiv & \Big[\Big(P(a) \wedge Q(a)\Big) \vee \Big(P(a) \wedge Q(b)\Big) \vee \Big(P(a) \wedge Q(c)\Big)\Big] \\
\bigwedge & \Big[\Big(P(b) \wedge Q(a)\Big) \vee \Big(P(b) \wedge Q(b)\Big) \vee \Big(P(b) \wedge Q(c)\Big)\Big] \\
\bigwedge & \Big[\Big(P(c) \wedge Q(a)\Big) \vee \Big(P(c) \wedge Q(b)\Big) \vee \Big(P(c) \wedge Q(c)\Big)\Big] \\
\equiv & \Big[P(a) \wedge \Big(Q(a) \vee Q(b) \vee Q(c)\Big)\Big] \\
\bigwedge & \Big[P(b) \wedge \Big(Q(a) \vee Q(b) \vee Q(c)\Big)\Big] \\
\bigwedge & \Big[P(c) \wedge \Big(Q(a) \vee Q(b) \vee Q(c)\Big)\Big]
\end{array}
$$

6. Let $I(x)$ be the statement that x has an internet connection, and $C(x, y)$ be the statement that x and y have chatted with each other over the internet. The universe of discourse for the variables x and y is the set of all students in your class. Express each of the following using logical operations and quantifiers.

   (a) Not everyone in your class has an internet connection.

   Solution:
   $$\neg\Big(\forall x I(x)\Big) \quad \text{or} \quad \exists x\Big(\neg I(x)\Big)$$

   (b) Everyone except one student in your class has an internet connection.

   Solution:
   $$\exists x\Big(\neg I(x) \wedge \forall y\big(\neg I(y) \rightarrow x = y\big)\Big)$$

   (c) Everyone in your class with an internet connection has chatted over the internet with at least one other student in your class.

   Solution:
   $$\forall x\Big(I(x) \rightarrow \exists y\big(C(x, y) \wedge x \neq y\big)\Big)$$

   (d) Someone in your class has an internet connection but has not chatted with anyone over the internet.

   Solution:
   $$\exists x\Big(I(x) \wedge \forall y\big(\neg C(x, y)\big)\Big)$$

7. For each part in the previous question, form the negation of the statement so that all negation symbols occur immediately in front of predicates. For example:

$$\neg\Big[\forall x\Big(P(x)\land Q(x)\Big)\Big] \;\equiv\; \exists x\Big[\neg\Big((P(x)\land Q(x)\Big)\Big] \;\equiv\; \exists x\Big[\Big(\neg P(x)\Big)\lor\Big(\neg Q(x)\Big)\Big]$$

(a) It is not so, that not everyone in your class has an internet connection.

Solution:

$$\neg\Big[\neg\Big(\forall x\, I(x)\Big)\Big] \;\equiv\; \forall x\, I(x)$$

Everyone in your class has an internet connection.

(b) It is not so, that everyone except one student in your class has an internet connection.

Solution:

$$\neg\Big[\exists x\,\Big(\neg I(x)\land\forall y\,(\neg I(y)\to x=y)\Big)\Big]$$

$$\equiv\;\forall x\,\neg\Big[\neg I(x)\land\forall y\,\Big(\neg I(y)\to x=y\Big)\Big]$$

$$\equiv\;\forall x\,\Big[\neg\neg I(x)\lor\neg\Big(\forall y\,\Big(\neg I(y)\to x=y\Big)\Big)\Big]$$

$$\equiv\;\forall x\,\Big[I(x)\lor\neg\Big(\forall y\,\Big(\neg\neg I(y)\lor x=y\Big)\Big)\Big]$$

$$\equiv\;\forall x\Big[I(x)\lor\exists y\,\neg\Big(I(y)\lor x=y\Big)\Big)\Big]$$

$$\equiv\;\forall x\Big[I(x)\lor\exists y\Big(\neg I(y)\land x\neq y\Big)\Big)\Big]$$

For all students, either the student has an internet connection, or there is another different student that does not have an internet connection
Either all students have an internet connection, or there are at least two students that do not have an internet connection.

(c) It is not so, that everyone in your class with an internet connection has chatted over the internet with at least one other student in your class.

Solution:

$$
\begin{aligned}
& \neg\Big[\forall x \left(I(x) \to \exists y \left(C(x,y) \land x \neq y\right)\right)\Big] \\
\equiv\;& \exists x \neg\Big[I(x) \to \exists y\Big(C(x,y) \land x \neq y\Big)\Big] \\
\equiv\;& \exists x \neg\Big[\neg I(x) \lor \exists y\Big(C(x,y) \land x \neq y\Big)\Big] \\
\equiv\;& \exists x \Big[I(x) \land \neg\Big(\exists y\Big(C(x,y) \land x \neq y\Big)\Big] \\
\equiv\;& \exists x \Big[\Big(I(x) \land \forall y \left(\neg C(x,y) \lor x = y\right)\Big] \\
\equiv\;& \exists x \Big[\Big(I(x) \land \forall y \left(C(x,y) \to x = y\right)\Big]
\end{aligned}
$$

There is a student with an internet connection, who hasn't chatted with anyone (except possibly himself).

(d) It is not so, that someone in your class has an internet connection but has not chatted with anyone over the internet.

Solution:

$$
\begin{aligned}
& \neg\Big[\exists x \left(I(x) \land \forall y \left(\neg C(x,y)\right)\right)\Big] \\
\equiv\;& \forall x \neg\Big[I(x) \land \forall y \left(\neg C(x,y)\right)\Big)\Big] \\
\equiv\;& \forall x \Big[\neg I(x) \lor \neg\forall y \left(\neg C(x,y)\right)\Big] \\
\equiv\;& \forall x \Big[\neg I(x) \lor \exists y \left(\neg\neg C(x,y)\right)\Big] \\
\equiv\;& \forall x \Big[\neg I(x) \lor \exists y \left(C(x,y)\right)\Big] \\
\equiv\;& \forall x \Big[I(x) \to \exists y \left(C(x,y)\right)\Big]
\end{aligned}
$$

All students who have an internet connection have chatted with someone (possibly themselves).

8. Determine the truth value of each of the following statements if the universe of discourse of each variable consists of all real numbers.

   (a) $\forall x \exists y (x + y = 1)$

   Solution: True.
   Let $y = 1 - x$.

   (b) $\exists x \exists y \Big((x + 2y = 2) \wedge (2x + 4y = 5) \Big)$

   Solution: False.
   If $x + 2y = 2$ then $2x + 4y = 4$ so $2x + 4y \neq 5$.

   (c) $\forall x \exists y \Big((x + y = 2) \wedge (2x - y = 1) \Big)$

   Solution: False.
   Let $x = 0$.

   (d) $\forall x \forall y \exists z (z = (x + y)/2)$

   Solution: True.
   For every pair of real numbers, there is a real number half-way in between them.