University of Ottawa - Department of Mathematics and Statistics
MAT 1322 D - Calculus II
Instructor: Petko Kitanov
October 8, 2014
Midterm Examination I

Name:....................................... Student Number:............

Instructions :

- Please write your name and student number on the indicated area above.
- This is a close book exam. It contains **6 questions**; there are 30 points in total.
- You can use non-programable and non-graphical calculators but no other aids are permitted.
- Clearly indicate the solution of each problem.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- Time allowed: 80 minutes.

GOOD LUCK!

Student Number :____________________, Final Grade : ___________ out of 30

Question	1	2	3	4	5	6
Grade						

Question 1. **a**) [5 pts] Why is $\displaystyle\int_0^5 \frac{x}{x^2-9}\,dx$ an improper integral?
If it is convergent, compute its value. If not, explain why it is divergent.

Solution. The integral is improper because the integrand $\dfrac{x}{x^2-9} = \dfrac{x}{(x-3)(x+3)}$ is undefined at the point $x = 3$, which is in the interval of integration $[0, 5]$. We have

$$\int_0^5 \frac{x}{x^2-9}dx = \int_0^3 \frac{x}{x^2-9}dx + \int_3^5 \frac{x}{x^2-9}dx$$

provided both integrals on the right hand side converge. Putting $u = x^2 - 9$, we find

$$\int \frac{x}{x^2-9}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2-9| + C.$$

So

$$\int_0^3 \frac{x}{x^2-9}dx = \lim_{t\to 3^-}\int_0^t \frac{x}{x^2-9}dx = \lim_{t\to 3^-}\frac{1}{2}(\ln|t^2-9| - \ln(9)) = -\infty.$$

Since this integral is divergent, the given integral is divergent also.

b) [5 pts] Determine if $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}+3x^2}\,dx$ is convergent or divergent using an appropriate comparison test.

Solution. For all $x \geq 1$, we have $\sqrt{x} \leq x^2$, so

$$3x^2 \leq \sqrt{x} + 3x^2 \leq 4x^2$$

whence

$$\frac{1}{4x^2} \leq \frac{1}{\sqrt{x}+3x^2} \leq \frac{1}{3x^2}$$

and it follows that

$$\frac{1}{4} = \int_1^\infty \frac{dx}{4x^2} \leq \int_1^\infty \frac{dx}{\sqrt{x}+3x^2} \leq \int_1^\infty \frac{dx}{3x^2} = \frac{1}{3}.$$

In particular the integral is convergent, and its value is 1/3.

Question 1'. **a**) [5 pts] Why is $\displaystyle\int_0^5 \frac{x}{x^2-4}\,dx$ an improper integral?
If it is convergent, compute its value. If not, explain why it is divergent.

Solution. The integral is improper because the integrand $\dfrac{x}{x^2-4} = \dfrac{x}{(x-2)(x+2)}$ is undefined at the point $x = 2$, which is in the interval of integration $[0, 5]$. We have

$$\int_0^5 \frac{x}{x^2-4}dx = \int_0^2 \frac{x}{x^2-4}dx + \int_2^5 \frac{x}{x^2-4}dx$$

provided both integrals on the right hand side converge. Putting $u = x^2 - 4$, we find

$$\int \frac{x}{x^2-4}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2-4| + C.$$

So

$$\int_0^2 \frac{x}{x^2-4}dx = \lim_{t\to 2^-}\int_0^t \frac{x}{x^2-4}dx = \lim_{t\to 2^-}\frac{1}{2}(\ln|t^2-4| - \ln(4)) = -\infty.$$

Since this integral is divergent, the given integral is divergent also.
b) [5 pts] Determine if $\displaystyle\int_1^\infty \frac{1}{2\sqrt{x}+x^2}\,dx$ is convergent or divergent using an appropriate comparison test.

Solution. for all $x \geq 1$, we have $\sqrt{x} \leq x^2$, so

$$x^2 \leq 2\sqrt{x} + x^2 \leq 3x^2$$

whence

$$\frac{1}{3x^2} \leq \frac{1}{2\sqrt{x}+x^2} \leq \frac{1}{x^2}$$

and it follows that

$$\frac{1}{3} = \int_1^\infty \frac{dx}{3x^2} \leq \int_1^\infty \frac{dx}{2\sqrt{x}+x^2} \leq \int_1^\infty \frac{dx}{x^2} = 1.$$

In particular the integral is convergent, and its value is 1.

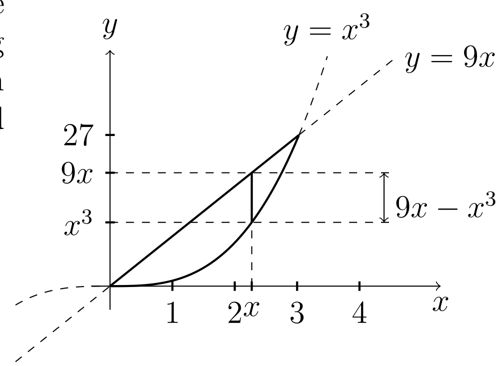
Question 2. [10 pts] Let $\mathcal{R}$ be the region in the first quadrant bounded by the curves

$$y = 9x \quad \text{et} \quad y = x^3 .$$

a) Sketch the region $\mathcal{R}$ and calculate its area.

Solution. We first determine the points of intersection of the two curves by solving the equation $9x = x^3$. We find $x = 0$ or $9 = x^2$, so $x = -3$, 0 or 3. Looking for those points in the first quadrant, we are left with $x = 0, 3$. So the points of intersection are $(0, 0)$ and $(3, 27)$.
The area of the region is

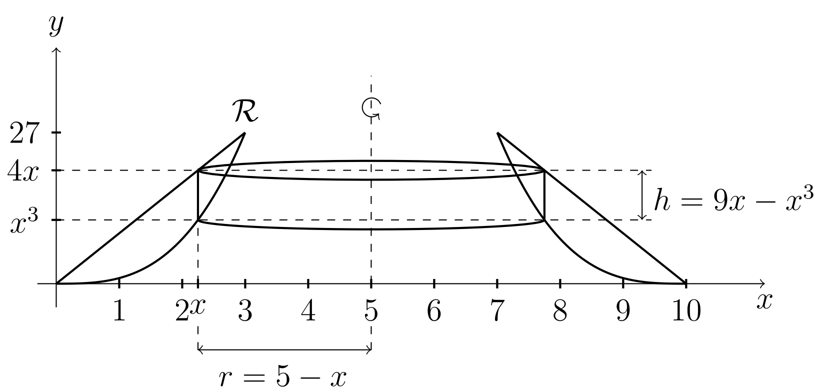
$$\int_0^3 (9x - x^3)\, dx = \left[\frac{9x^2}{2} - \frac{x^4}{4} \right|_0^3 = \boxed{\frac{81}{4}}.$$



b) Calculate the volume of the solid of revolution obtained by rotating the region $\mathcal{R}$ around the vertical line $x = 5$, and draw a typical layer which appears in the calculation of volume (disc, washer or cylindrical shell), labeling its dimensions.

Solution. The easiest technique is the method of cylindrical shells. Rotating the portion of the region $\mathcal{R}$ between x and $x + \Delta x$ (for small Δx) about the line $x = 5$, we obtain a cylindrical shell of radius $r = 5 - x$, of height $h = 9x - x^3$ and thickness Δx (as shown in the figure). Its volume is

$$\Delta V \cong 2\pi r h \Delta x = 2\pi(5 - x)(9x - x^3)\Delta x$$



Thus, the total volume of the solid is

$$V = \int_0^3 2\pi(5 - x)(9x - x^3)dx = 2\pi \int_0^3 (45x - 9x^2 - 5x^3 + x^4)dx$$

$$= 2\pi \left[(45/2)x^2 - 3x^3 - (5/4)x^4 + (1/5)x^5 \right|_0^3 = \frac{1377\pi}{10} \cong 432.6\, .$$

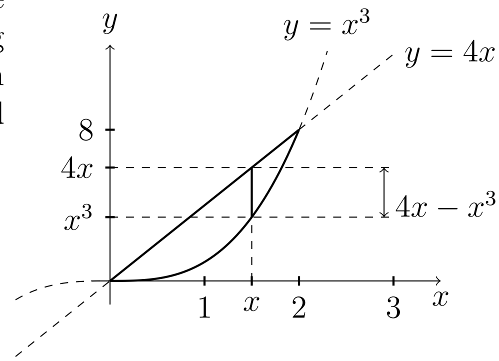
Question 2'. [10 pts] Let $\mathcal{R}$ be the region in the first quadrant bounded by the curves

$$y = 4x \quad \text{and} \quad y = x^3 .$$

a) Sketch the region $\mathcal{R}$ and calculate its area.

Solution. We first determine the points of intersection of the two curves by solving the equation $4x = x^3$. We find $x = 0$ or $4 = x^2$, so $x = -2$, 0 or 2. Looking for those points in the first quadrant, we are left with $x = 0, 2$. So the points of intersection are $(0, 0)$ and $(2, 8)$.
The area of the region is

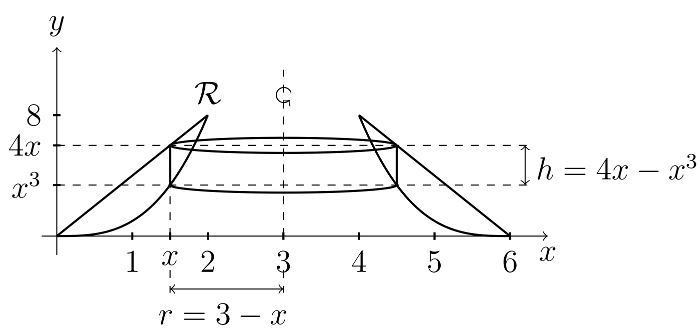
$$\int_0^2 (4x - x^3)\, dx = \left[2x^2 - \frac{x^4}{4} \right|_0^2 = \boxed{4}.$$



b) Calculate the volume of the solid of revolution obtained by rotating the region $\mathcal{R}$ around the vertical line $x = 3$, and draw a typical layer which appears in the calculation of volume (disc, washer or cylindrical shell), labeling its dimensions.

Solution. The easiest technique is the method of cylindrical shells. Rotating the portion of the region $\mathcal{R}$ between x and $x + \Delta x$ (for small Δx) about the line $x = 3$, we obtain a cylindrical shell of radius $r = 3 - x$, with height $h = 4x - x^3$ and thickness Δx (as shown in the figure). Its volume is

$$\Delta V \cong 2\pi r h \Delta x = 2\pi(3 - x)(4x - x^3)\Delta x$$

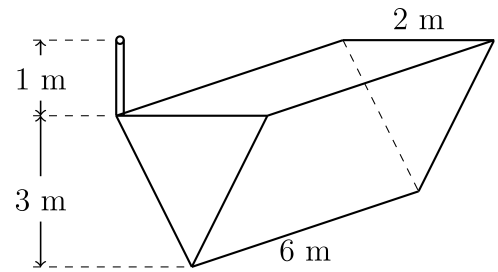


Thus, the total volume of the solid is

$$V = \int_0^2 2\pi(3 - x)(4x - x^3)dx = 2\pi \int_0^2 (12x - 4x^2 - 3x^3 + x^4)dx$$
$$= 2\pi \left[6x^2 - (4/3)x^3 - (3/4)x^4 + (1/5)x^5 \right|_0^2 = \frac{232\pi}{15} \cong 48.59.$$

Question 3. [10 pts] A reservoir in the form of a straight prism with triangular base is shown in the figure to the right.

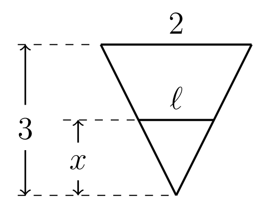
Its vertical faces are isosceles triangles of height 3 m and base 2 m, its length is 6 m, it is near the surface of the Earth, and it is full of water, which will be pumped to a height of 1 m above the reservoir.



Denote by x the height in meters measured **from the bottom of the reservoir**.

a) What is, at first approximation, the volume of the layer of water between the heights x and $x + \Delta x$?

Solution. A horizontal section of the reservoir at height x is a rectangle of length 6 and width ℓ where ℓ is given by examination of similar triangles (figure to the right). We find



$$\frac{\ell}{x} = \frac{2}{3} \quad \Longrightarrow \quad \ell = \frac{2x}{3}.$$

$$\Longrightarrow \quad \text{(volume of the layer)} \cong \text{(area of rectangle)} \times \Delta x = 6\ell\Delta x = 4x\Delta x.$$

b) What is, at first approximation, the work required to pump that layer of water to a height of 1 m above the reservoir? Recall that the density of water is 1000 kg/m^3, and gravitational acceleration at the surface of the earth is $g \cong 9.8$ m/s^2.

Solution. We must lift the layer up by $4 - x$ metres. So the required work is

$$\begin{aligned} \text{Work} &= \text{(Force)} \times \text{(distance)} \\ &= 1000g\text{(volume)} \times \text{(distance)} \\ &\cong 9800(4x\Delta x)(4 - x) = 39200x(4 - x)\Delta x \end{aligned}$$

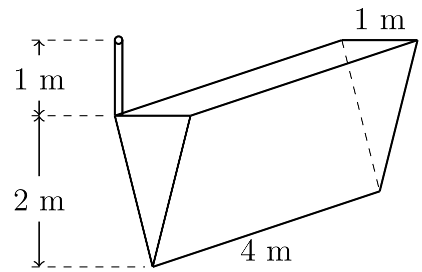
c) What is, in Joules, the work required to pump all the water from the reservoir to a height of 1 m above the reservoir?

Solution. Since the reservoir is full of water, we integrate from $x = 0$ to $x = 3$:

$$W = \int_0^3 39200x(4 - x)dx = 39200 \int_0^3 (4x - x^2)dx = 39200 \left[2x^2 - \frac{1}{3}x^3\right|_0^3 \cong 352800 \text{ J.}$$

Question 3'. [10 pts] A reservoir in the form of a straight prism with triangular base is shown in the figure to the right.

Its vertical faces are isosceles triangles of height 2 m and base 1 m, its length is 4 m, it is near the surface of the Earth, and it is full of water, which will be pumped to a height of 1 m above the reservoir.

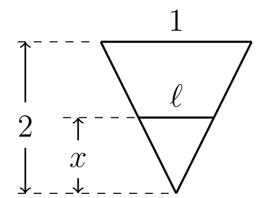


Denote by x the height in meters measured **from the bottom of the reservoir**.

a) What is, at first approximation, the volume of the layer of water between the heights x and $x + \Delta x$?

Solution. A horizontal section of the reservoir at height x is a rectangle of length 4 and width ℓ where ℓ is given by examination of similar triangles (figure to the right). We find

$$\frac{\ell}{x} = \frac{1}{2} \quad \text{so} \quad \ell = \frac{x}{2}.$$



So, (volume of the layer) $\cong$ (area of rectangle) $\times \Delta x = 4\ell\Delta x = 2x\Delta x$.

b) What is, at first approximation, the work required to pump that layer of water to a height of 1 m above the reservoir? Recall that the density of water is 1000 kg/m^3, and gravitational acceleration at the surface of the earth is $g \cong 9.8$ m/s^2.

Solution. We must lift the layer up $3 - x$ meters. So the required work is

$$\begin{aligned} \text{Work} &= (\text{Force}) \times (\text{distance}) \\ &= 1000g(\text{volume}) \times (\text{distance}) \\ &\cong 9800(2x\Delta x)(3-x) = 19600x(3-x)\Delta x \end{aligned}$$

c) What is, in Joules, the work required to pump all the water from the reservoir to a height of 1 m above the reservoir?

Solution. Since the reservoir is full of water, we integrate from $x = 0$ to $x = 2$:

$$W = \int_0^2 19600x(3-x)dx = 19600\int_0^2 (3x - x^2)dx = 19600\left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right|_0^2 \cong 65300 \text{ J}.$$

Question 4. [6 points]
a) Consider the curve $x = t^2\,,\ y = e^{2t}\,,\quad 1 \le t \le 3.$
Write down the integral which gives the length of this curve. Simplify the integrand, but *do not calculate the integral.*

Response:

$$L = \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^3 \sqrt{(2t)^2 + (2e^{2t})^2}\,dt = \int_1^3 \sqrt{4t^2 + 4e^{4t}}\,dt\,.$$

b) In a city, the temperature (in °C) t hours after 9 am is approximately given by the function

$$T(t) = 10 + \frac{70}{9}\sin\frac{\pi t}{12}$$

Construct the integral which will compute the average temperature between 9 am and 9 pm. *do not calculate the integral.*

Response: Let $t = 0$ when it is 9 am, thus $t = 12$ when it is 9 pm. Therefore, the average value is given by

$$avg(T) = \frac{1}{b-a}\int_a^b T(t)\,dt = \frac{1}{12}\int_0^{12}\left(10 + \frac{70}{9}\sin\frac{\pi t}{12}\right)dt$$

Question 4'. [6 points]
a) Consider the curve $x = t^2\,,\ \ y = e^{3t}\,,\qquad 1 \le t \le 4.$
Write down the integral which gives the length of this curve. Simplify the integrand, but *do not calculate the integral.*

Response:

$$L = \int_1^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^4 \sqrt{(2t)^2 + (3e^{3t})^2}\,dt = \int_1^4 \sqrt{4t^2 + 9e^{6t}}\,dt\,.$$

b) In a city, the temperature (in °C) t hours after 9 am is approximately given by the function

$$T(t) = 10 + \frac{70}{9}\sin\frac{\pi t}{12}$$

Construct the integral which will compute the average temperature between 9 am and 5 pm. *do not calculate the integral.*

Response: Let $t = 0$ when it is 9 am, thus $t = 8$ when it is 5 pm. Therefore, the average value is given by

$$avg(T) = \frac{1}{b-a}\int_a^b T(t)\,dt = \frac{1}{8}\int_0^8 \left(10 + \frac{70}{9}\sin\frac{\pi t}{12}\right)dt$$

5. (a) [2 points] Match the differential equation $y' = 3 + x - y$ with its direction field and sketch the graph of the solution that satisfy the initial condition $y(0) = 3$.

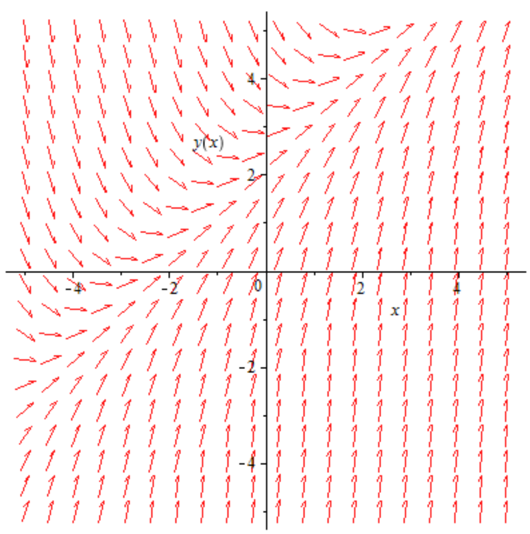


Fig. A

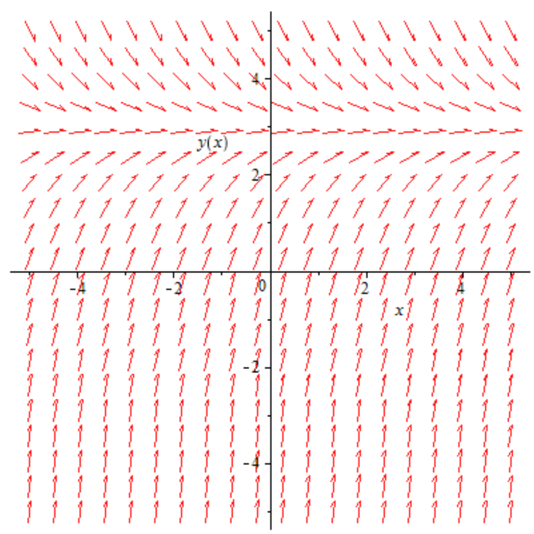


Fig. B

(b) [4 points] Use Euler's method with step $h = 0.1$ to find the first two approximations (y_1 and y_2) of the initial value problem

$$y' = 3 + x - y, \quad y(0) = 3.$$

Solution. (a) The correct answer is Fig. A.
(b) The formula for the approximations is

$$y_n = y_{n-1} + hf\left(x_{n-1}, y_{n-1}\right)$$

So, we have

$$y_1 = y_0 + (0.1)(3 + x_0 - y_0) = 3 + (0.1)(3 + 0 - 3) = 3 + (0.1)(0) = 3$$

$$y_2 = y_1 + (0.1)(3 + x_1 - y_1) = 3 + (0.1)(3 + 0.1 - 3) = 3 + (0.1)(0.1) = 3.01$$

5'. (a) [2 points] Match the differential equation $y' = 1 + x + y$ with its direction field and sketch the graph of the solution that satisfy the initial condition $y(0) = 2$.

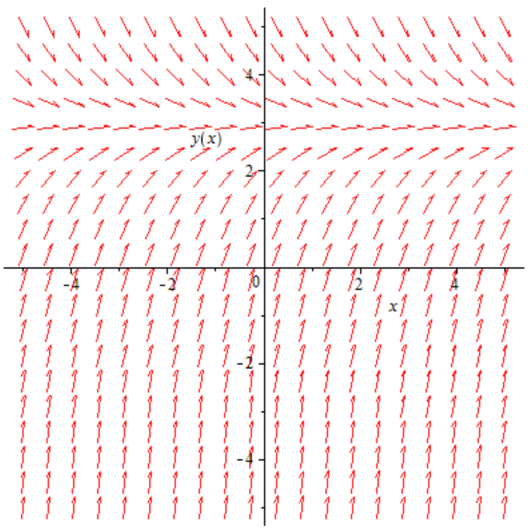


Fig. A

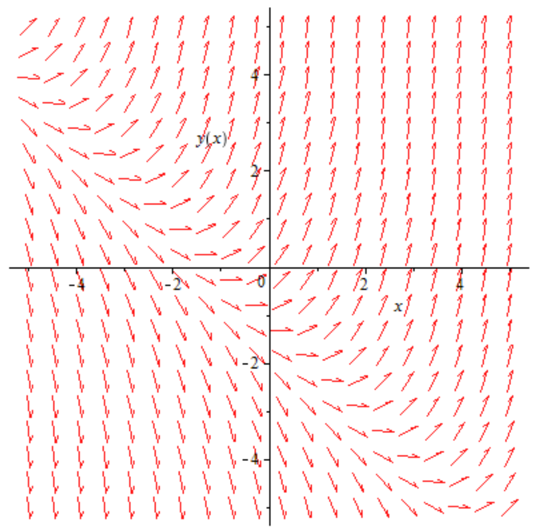


Fig. B

(b) [4 points] Use Euler's method with step $h = 0.1$ to find the first two approximations (y_1 and y_2) of the initial value problem

$$y' = 1 + x + y, \quad y(0) = 2.$$

Solution. (a) The correct answer is Fig. B.
(b) The formula for the approximations is

$$y_n = y_{n-1} + hf\left(x_{n-1}, y_{n-1}\right)$$

So, we have

$$y_1 = y_0 + (0.1)(1 + x_0 + y_0) = 2 + (0.1)(1 + 0 + 2) = 2 + (0.1)(3) = 2.3$$

$$y_2 = y_1 + (0.1)(1 + x_1 + y_1) = 2.3 + (0.1)(1 + 0.1 + 2.3) = 2.3 + (0.1)(3.4) = 2.64$$

6. (a) [4 points] Solve the differential equation $y' + y^2 \sin x = 0$. Write down the solution in explicit form

(b) [4 points] Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{\ln x}{xy}, \quad y(1) = 2$

Solution. (a) Separating the variables, we get

$$\frac{dy}{y^2} + \sin x dx = 0$$

Integrating

$$\int \frac{dy}{y^2} + \int \sin x dx = C$$

we get the solution

$$-\frac{1}{y} - \cos x = C.$$

The explicit form of the solution is

$$y = -\frac{1}{\cos x + C}.$$

(b) First we find the general solution by separating the variables

$$y dy = \frac{\ln x}{x} dx$$

$$\int y dy = \int \frac{\ln x}{x} dx + C$$

$$\frac{y^2}{2} = \frac{ln^2 x}{2} + C$$

or

$$y^2 = ln^2 x + C$$

The initial condition yields $4 = C$. So the solution of the IVP is

$$y^2 = \ln^2 x + 4.$$

6'. (a) [4 points] Solve the differential equation $y' - 2x(1 + y^2) = 0$. Write down the solution in explicit form

(b) [4 points] Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{2x}{1 + 2y}, \quad y(2) = 0$

Solution. (a) Separating the variables, we get

$$\frac{dy}{1 + y^2} - 2x dx = 0$$

Integrating

$$\int \frac{dy}{1+y^2} - \int 2x dx = C$$

we get the solution

$$\arctan y - x^2 = C.$$

The explicit form of the solution is

$$y = \tan(x^2 + C).$$

(b) First we find the general solution by separating the variables

$$(1+2y)dy = 2xdx$$

$$\int (1+2y)dy = \int 2xdx + C$$

$$y + y^2 = x^2 + C$$

The initial condition yields $0 = 4 + C$ or $C = -4$. So the solution of the IVP is

$$y + y^2 = x^2 - 4.$$