MAT2377

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Chapter 3

Comments

- These notes cover material from Chapter 3, Sections 3.1-3.2, 3.4-3.9, 3.11. Sections 3.3, 3.10, 3.13 will not be discussed and will be included in assignments or exams. Section 3.12 will be discussed later.

- I'm not re-writing the textbook.

- In class I do some examples that are not included in the notes and vice versa: here you can find some examples that I do not do in class.

- There may be some typos. The final version of the slides will be posted *after* the chapter is finished.

Binomial Coefficient

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}\,.$$

- $2! \times 4! = (1 \times 2) \times (1 \times 2 \times 3 \times 4) = 48$, but $(2 \times 4)! = 8! = 40320$.
- $\binom{5}{1} = \frac{5!}{1! \times 4!} = \frac{1 \times 2 \times 3 \times 4 \times 5}{1 \times (1 \times 2 \times 3 \times 4)} = \frac{5}{1} = 5$; In general: $\binom{n}{1} = n$. Also $\binom{n}{0} = 1$
- $\binom{6}{2} = \frac{6!}{2! \times 4!} = \frac{4! \times 5 \times 6}{2! \times 4!} = \frac{5 \times 6}{2} = 15$
- $\binom{27}{22} = \frac{27!}{22! \times 5!} = \frac{22! \times 23 \times 24 \times 25 \times 26 \times 27}{5! \times 22!} = \frac{23 \times 24 \times 25 \times 26 \times 27}{120}$

Binomial experiment

- A Bernoulli trial is a random experiment with two possible outcomes, "success" and "failure". Let p denote the probability of a success.

- A binomial experiment consists of n repeated independent Bernoulli trials, each with the same probability of success, p.

Examples:

- female/male births; satisfactory/defective items on prod'n line; sampling with replacement with two types of item.

Binomial distribution

If we have a binomial experiment (i.e. n independent events, each with prob p) then the number X of them that occur is a random variable with a binomial(n, p) distribution:

$$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad \text{for } x = 0, 1, 2, \ldots, n.$$

This often abbrev'd to "$X \sim \mathcal{B}(n, p)$". (X is binomial rv or has binomial distribution with parameters n and p)

- If $X \sim B(1, p)$ then $P(X = 0) = 1 - p$, $P(X = 1) = p$ and so

$$\mathrm{E}(X) = (1-p) \times 0 + p \times 1 = p\,.$$

Expectation and Variance for Binomial

Recall: if $X \sim \mathcal{B}(n,p)$ then $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, 2, \ldots, n$. It can be shown that

$$\mathrm{E}(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \cdots = \boldsymbol{np}$$

and so

$$\mathrm{Var}(X) = \mathrm{E}\left[(X - np)^2\right] = \sum_{x=0}^{n} (x - np)^2 \binom{n}{x} p^x (1-p)^{n-x} = \boldsymbol{np(1-p)}\,.$$

Later we will see an easier way to derive these by interpreting X as a sum of other rv's.

Example :

Suppose that each sample of water has 10% of being polluted. If 12 samples are selected independently, then it is reasonable to model the number X polluted sample as in the sample as $\mathcal{B}(12, 0.1)$. Find

(a) $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(b) $P(X = 3)$.

(c) $P(X \leq 3)$ (tables can be used).

(a) If $X \sim \mathcal{B}(n,p)$ then $\mathrm{E}(X) = np$ and $\mathrm{Var}(X) = np(1-p)$ so

$$\mathrm{E}(X) = 12 \times 0.1 = 1.2 \quad ; \quad \mathrm{Var}(X) = 12 \times 0.1 \times 0.9 = 1.08\,.$$

(b) $P(X = 3) = \binom{12}{3}(0.1)^3(0.9)^9 \approx 0.0852.$

(c) $P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = \cdots.$ However, $P(X \le 3)$ for $X \sim B(12, 0.1)$ is tabulated in the textbook: $\approx 0.9744.$

- Also, since $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2) \approx 0.8891$ from tables we see that $P(X = 3) = P(X \le 3) - P(X \le 2) \approx 0.9744 - 0.8891 = 0.0853$

Example:

- An airline sells 101 tickets for a flight with 100 seats. Each ticketed passenger is known to have a 0.97 pb. of showing up for flight. What is probability of 101 passengers showing up (and airline caught overbooking)? Make appropriate assumptions.

- Answer the same if the airline sells 125 tickets.

Geometric random variable

Consider a sequence of Bernoulli trials, with probability p of success at each step. The geometric random variable X denote the number of steps before the first success occurs. The probability distribution is given by

$$f(x) = P(X = x) = (1 - p)^{x-1}p, \quad x = 1, \dots$$

We will write $X \sim Geo(p)$. For this random variable we have $E(X) = \frac{1}{p}$.

Example: A fair die is thrown until it shows six. What is the probability that 5 throws are required?

Solution: If 5 throws are required, we have to compute $P(X = 5)$, where X is geometric $Geo(1/6)$.

Poisson random variable

Let the number of “changes” that occur in a continuous interval (of time or space) be counted. We have a Poisson process of rate λ if:

(a) The number of changes occurring in nonoverlapping intervals are independent.

(b) The probability of exactly one change in a short interval of length h is approximately $\lambda \times h$.

(c) The probability of two or more changes in a sufficiently short interval is essentially 0.

Assume that an experiment satisfies the above properties. Let X be the number of changes in an interval of length 1. How to calculate p.m.f of X?

We partition the unit interval into n disjoint subintervals of length $1/n$. By condition (b) the probability of one change occurring in one small subinterval is approximately $\lambda \times 1/n$. By condition (c) the probability of two or more changes is essentially equal to 0. By condition (a) we have a sequence of n Bernoulli trials with probability $p = \lambda \times 1/n$. Therefore

$$f(x) = P(X = x) \approx \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

Letting $n \to \infty$ we get

$$P(X = x) = \exp(-\lambda)\frac{\lambda^x}{x!}$$

Let $X \sim$ Poisson, λ:

- The p.m.f. of X is

$$f(x) = P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!},\ x = 0, \ldots, \infty$$

- $\mathrm{E}(X) = \mu = \lambda$

- $\mathrm{Var}(X) = \sigma^2 = \lambda$

Example:

A traffic flow is typically modeled by a Poisson distribution. It is known that the traffic flowing through an intersection is 6 cars/minute on average.

1. What is the probability of no cars through the intersection within 30 seconds?

Answer: 6 cars/min=3 cars/30 sec.. Thus $\lambda = 3$. To compute $P(X = 0)$.

Example

Here's an example where the Poisson distribution was used in a maternity hospital to work out how many births would be expected during the night. The hospital had 3000 deliveries each year, so if these happened randomly around the clock 1000 deliveries would be expected between the hours of midnight and 8.00 a.m. This is the time when many staff are off duty and it is important to ensure that there will be enough people to cope with the workload on any particular night. The average number of deliveries per night is 1000/365, which is 2.74. From this average rate the probability of delivering 0,1,2, etc babies each night can be calculated using the Poisson distribution. Some probabilities are:

$$
\begin{array}{lll}
f(0) & 2.74^0 \exp(-2.74) \; / \; 0! & = 0.065 \\
f(1) & 2.74^1 \exp(-2.74) \; / \; 1! & = 0.177 \\
f(2) & 2.74^2 \exp(-2.74) \; / \; 2! & = 0.242 \\
f(3) & 2.74^3 \exp(-2.74) \; / \; 3! & = 0.221
\end{array}
$$

- On how many days in the year would 5 or more deliveries be expected? (Ans. 52)

- Over the course of one year, what is the greatest number of deliveries expected in any night? (Ans. 8)

- Why might the pattern of deliveries not follow a Poisson distribution? (Ans. If deliveries were not random throughout the 24 hours; e.g. if a lot of women had elective caesareans done during the day). Note: In this real life example, deliveries in fact followed the Poisson distribution very closely, and the hospital was able to predict the workload accurately.

Answer:

- If X is the number of babies during a night, then X has Poisson distribution with $\lambda = 2.74$. To compute $365 \times P(X \geq 5)$.

Simulation

- To simulate n random variables from a binomial distribution with parameters m and p, type `X=rbinom(n,m,p)`. For example

```
X=rbinom(1000,10,0.5); Y=rbinom(1000,10,0.1);
par(mfrow=c(1,2));
hist(X); hist(Y)
```

  You can also calculate $P(X \leq x)$, when X is binomial with parameters m and p, e.g. $x = 7$, $m = 10$, $p = 0.5$

```
dbinom(7,10,0.5); pbinom(7,10,0.5)
```

- To simulate n random variables from a Poisson distribution with parameter λ, type `X=rpoi(n,lambda)`. For example

```
X=rpois(1000,10); hist(X); mean(X)
```

Normal Distribution

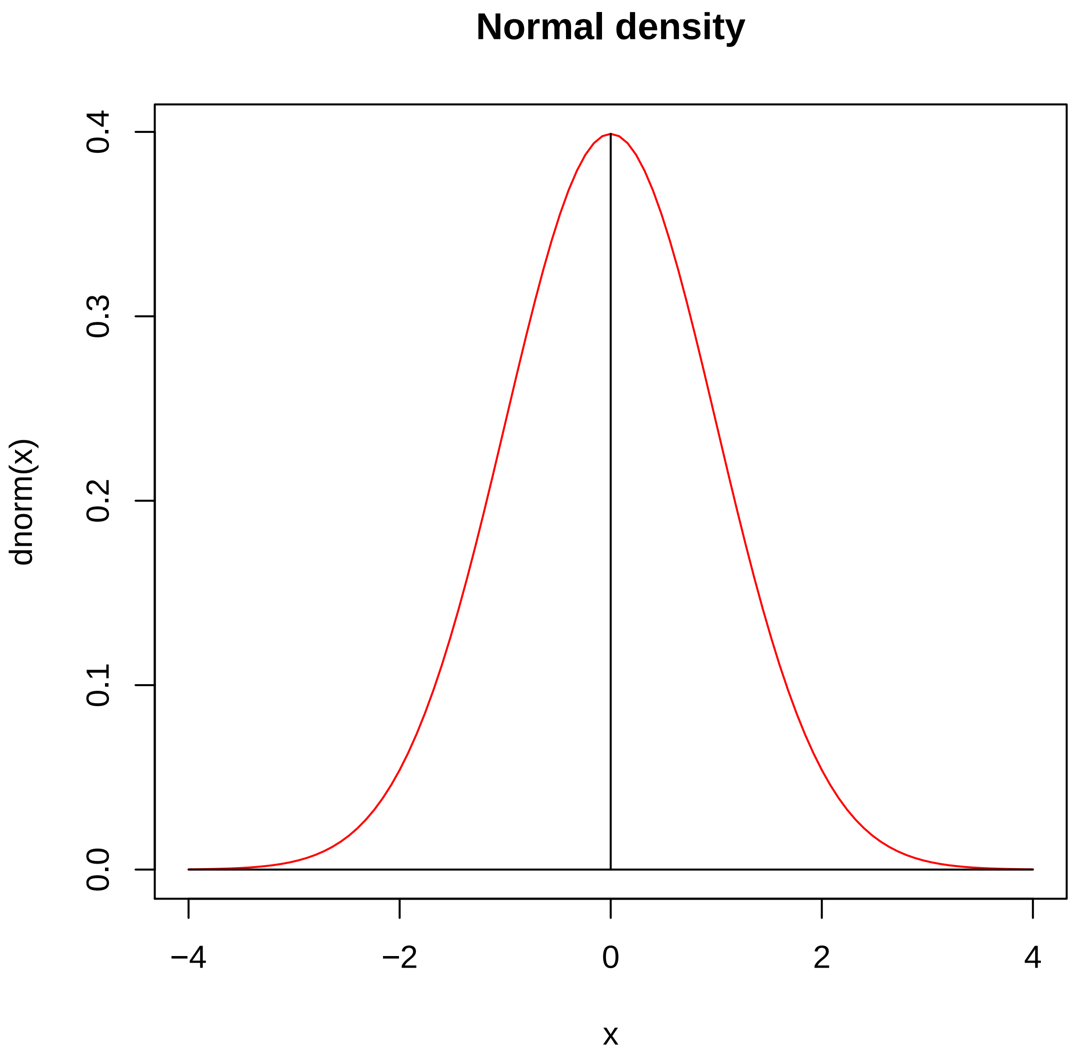
An important example is afforded by the special PDF

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} .$$

The corresponding CDF is denoted by

$$\Phi(x) = \int_{-\infty}^{x} \phi(z)\, dz .$$

A rv with this CDF is said to have a **standard normal distribution**. Such a rv is traditionally denoted (where sensible) by Z, and we write $Z \sim \mathcal{N}(0, 1)$.


Normal density
dnorm(x)
0.0
0.1
0.2
0.3
0.4
−4
−2
0
2
4
x

Standard Normal Random Variable

The expectation and variance of $Z \sim \mathcal{N}(0,1)$ are

$$\mathrm{E}(Z) = \int_{-\infty}^{\infty} z \, \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \, dz = 0,$$

$$\int_{-\infty}^{\infty} z^2 \phi(z) \, dz = 1$$

Hence if $Z \sim \mathcal{N}(0,1)$ then

$\mathrm{E}(Z) = 0$ and $\mathrm{Var}(Z) = 1 = SD(Z)$.

The general normal RV

Take a positive σ and any real μ. Then if $Z \sim \mathcal{N}(0,1)$ and $X = \mu + \sigma Z$ then

$$\frac{X-\mu}{\sigma} = Z \sim \mathcal{N}(0,1)$$

but the CDF of X is given by

$$F_X(x) = P\{X \le x\} = P\{\mu + \sigma Z \le x\} = P\left\{Z \le \frac{x-\mu}{\sigma}\right\} = \Phi\left(\frac{x-\mu}{\sigma}\right) .$$

The PDF is then

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}\Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)$$

A rv X with this CDF/PDF has

$$\mathrm{E}(X) = \mu + \sigma \mathrm{E}(Z) = \mu \quad , \quad \mathrm{Var}(X) = \sigma^2 \mathrm{Var}(Z) = \sigma^2 \;\Rightarrow\; \mathrm{SD}(X) = \sigma$$

and is said to be **Normal with Mean (expectation) $\boldsymbol{\mu}$ and Variance $\boldsymbol{\sigma^2}$**; write $X \sim \mathcal{N}(\mu, \sigma^2)$.

Examples

1. Assume that Z represents the standard normal random variable. Evaluate the following probabilities:

   (a) $P(Z \leq 0.5) = 0.6915$
   (b) $P(Z < -0.3) = 0.3821$
   (c) $P(Z > 0.5) = 1 - P(Z \leq 0.5) = 1 - 0.6915 = 0.3085$,
   (d) $P(0.1 < Z < 0.3) = P(Z < 0.3) - P(Z < 0.1) = 0.6179 - 0.5398 = 0.0781$,
   (e) $P(-1.2 < Z < 0.3) = P(Z < 0.3) - P(Z < -1.2) = 0.5028$.

2. Suppose that the waiting time (in minutes) for a coffee at 9am is normally distributed with mean 5 and standard deviation 0.5. What is the probability that one such waiting time is at most 6 minutes?

$Solution$: Let X denote the waiting time; then $X \sim \mathcal{N}(5, 0.5^2)$ and the *standardised rv*

$$Z = \frac{X - 5}{0.5} \sim \mathcal{N}(0, 1) \, .$$

The desired probability is

$$P\left\{X \le 6\right\} = P\left\{\frac{X-5}{0.5} \le \frac{6-5}{0.5}\right\} = P\left(Z \le \frac{6-5}{0.5}\right) = \Phi\left(\frac{6-5}{0.5}\right) = \Phi(2)$$

Reading from the tables, $\Phi(2) \approx 0.9772 \, .$

3. Suppose that bottles of beer are filled in such a way that the actual volume in them (in mL) varies randomly according to a normal distribution with mean 376.1 with standard deviation 0.4. What is the probability that the volume is less than 375mL?

Solution: Let X denote the volume; then $X \sim \mathcal{N}(376.1, 0.4^2)$ and so

$$Z = \frac{X - 376.1}{0.4} \sim \mathcal{N}(0,1)\,.$$

The desired probability is

$$P\left(X < 375\right) = P\left(\frac{X - 376.1}{0.4} < \frac{375 - 376.1}{0.4}\right) = P\left(Z < \frac{-1.1}{0.4}\right)\,.$$

So

$$\Phi\left(-2.75\right) = 0.003\,.$$

Reading table backwards

In some examples we are asked to go in the other direction: If $Z \sim \mathcal{N}(0,1)$, for which values a, b and c do we have

- $P(Z \leq a) = 0.95$;
- $P(|Z| \leq b) = P(-b \leq Z \leq b) = 0.99$;
- $P(|Z| \geq c) = 0.01$.

Solution:

- Observing the table we see that

$$P(Z \leq 1.64) \approx 0.9495 \quad \text{and} \quad P(Z \leq 1.65) \approx 0.9505\,.$$

Clearly we must have $1.64 < a < 1.65$; linearly interpolating, our best guess would be $a \approx 1.645$, although this level of precision is usually not necessary. Often the inital interval estimate is enough.

- Next, note that

$$P\left(-b \leq Z \leq b\right) = P(Z \leq b) - P(Z < -b)$$

However by symmetry etc. we get that

$$P(Z < -b) = P(Z > +b) = 1 - P(Z \leq +b)$$

and so

$$P(-b \leq Z \leq b) = P(Z \leq b) - [1 - P(Z \leq b)] = 2P(Z \leq b) - 1$$

If this is 0.99, then

$$P(Z \leq b) = \frac{1 + 0.99}{2} = 0.995\,;$$

Consulting the table we see that

$$P(Z \leq 2.57) \approx 0.9949 \quad \text{and} \quad P(Z \leq 2.58) \approx 0.9951$$

so again an interpolation suggests taking $b \approx 2.575$.

- Finally note that $\{|Z| \geq c\} = \{|Z| < c\}^c$ so we need c such that

$$P(|Z| < c) = 1 - P(|Z| \geq c) = 0.99$$

But this is the same as

$$P(-c < Z < c) = P(-c \leq Z \leq c) = 0.99$$

since $|x| < y \Leftrightarrow -y < x < y$, and $P(Z = c) = 0$ for all c. Hence again we take $c \approx 2.575$.

Exponential random variable

Assume that cars arrive according to a Poisson process with rate λ, i.e. number of cars within a particular time period is a Poisson random variable, say N, with parameter λ. Let X be the time to the first car arrival. Then

$$P(X > x) = P(N = 0) = \exp(-\lambda x).$$

We say that X has the exponential distribution with parameter λ, $X \sim$ exponential, λ.

- $$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda x} & \text{for } 0 \leq x \end{cases}$$

- $$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } 0 < x \\ 0 & \text{for } x \leq 0 \end{cases}$$

- $\mu = \mathrm{E}(X) = 1/\lambda$

- $\sigma^2 = \mathrm{Var}(X) = 1/\lambda^2$

- "Lack of memory":

$$P(X > s + t \mid X > t) = P(X > s)$$

Example: The lifetime of a certain type of lightbulb has an exponential distribution with mean 100 hours.

1. What is the probability that a lightbulb will last at least 100 hours?

2. Given that a lightbulb has already been burning for 100 hours, what is the probability that it will last at least 100 hours more?

3. The manufacturer wants to guarantee that his lightbulbs will last at least t hours. What should t be in order to ensure that 90% of the lightbulbs will last longer than t hours?

Answer: X is exponential with $\lambda = 1/100$. To compute $P(X > 100)$; $P(X > 200|X > 100) = P(X > 100)$; to find t such that $P(X > t) = 0.9$.

Gamma random variable

Assume that cars arrive according to a Poisson process with rate λ. Recall that if X is the time to the first car arrival, then X has exponential distribution with parameter λ. Now, if X is the time to the rth arrival, then X has **Gamma distribution** with parameters λ and r. We have

- $$\mu = \mathrm{E}(X) = \frac{r}{\lambda}$$
- $$\sigma^2 = \mathrm{Var}(X) = \frac{r}{\lambda^2}$$

Normal approximation to the binomial

If $X \sim \mathcal{B}(n,p)$ then *we may interpret* X as a sum of iid rv's:

$$X = I_1 + I_2 + \cdots + I_n \text{ where each } I_i \sim \mathcal{B}(1,p)\,.$$

Thus according to the Central Limit Theorem, for large n, the standardised version

$$\frac{X - np}{\sqrt{np(1-p)}} \overset{\text{approx}}{\sim} \mathcal{N}(0,1)\,,$$

i.e. for large n if $X \overset{\text{exact}}{\sim} \mathcal{B}(n,p)$ then $X \overset{\text{approx}}{\sim} \mathcal{N}(np, np(1-p))$.

This means that binomial probabilities can be approximated by normal probabilities.