MAT2377

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Chapter 2

Comments

- These notes cover material from Chapter 2, Sections 2.1-2.3, 2.5-2.7. **In class, I will use a blackboard**.
- I'm not re-writing the textbook.
- In class I do some examples that are not included in the notes and vice versa: here you can find some examples that I do not do in class.
- There may be some typos. The final version of the slides will be posted *after* the chapter is finished.

Discrete Random Variables

Random Variable is **a numerical outcome an experiment**.

If we have a probability rule also defined on the experiment, the prob. that the rv takes each of its values is called its **probability distribution** or **probability function** or **probability mass function**.

Notation for random variables (rv's)

- Capital Roman letters e.g. X, Y used to denote rv's.

- Corresponding lower case letters e.g. x, y used to denote *generic values taken* by rv.

- A rv is a way to define events: if X takes values $0, 1, 2, \ldots$ then we can define events $\{X = 0\}$, $\{X = 1\}$, $\{X = 2\}$,. . . etc.

- The probability distribution (mass) function is

$$f(x) = P(\{X = x\}) = P(X = x) = \ldots$$

- Cumulative distribution function

$$F(x) = P(X \leq x)$$

Example:

1. Flip a fair coin. Assign $X = 1$ if Head occurs, $X = 0$ if Tail occurs. Then $P(X = 1) = \frac{1}{2}$, $P(X = 0) = \frac{1}{2}$ is the probability distribution function.

2. Roll 1 fair die. X is the number. Then $P(X = 1) = P(X = 2) = \cdots = P(X = 6) = \frac{1}{6}$ and $P(3 \leq X \leq 5) = P(X = 3) + P(X = 4) + P(X = 5) = \frac{1}{2}$.

3. Number of calls, X, is a random variable with possible values $0, 1, 2, \ldots$.

Expected value (mean)

We define the mean of a discrete random variable X as

$$\mu = \mathrm{E}(X) = \sum_{x} x P(X = x)\,,$$

where the sum extends over all values x the rv X takes.

The expectation is just a property of a probability distribution, but we can *interpret* it as a “long-run average”.

Examples

- Expected roll of fair die: $P(Z=1)=P(Z=2)=\cdots=P(Z=6)=\frac{1}{6}$,

$$\mathrm{E}(Z)=\frac{1}{6}\times 1+\frac{1}{6}\times 2+\cdots=\frac{1}{6}(1+2+3+4+5+6)=\frac{21}{6}=3.5\,.$$

- In a gambling game a player can win wins $\$3$ with probability $\frac{1}{3}$ and lose $\$1$ with probability $\frac{2}{3}$ for each $\$1$ bet. Find expected value of the game. Hint: value of the game = win - bet.

  Solution: Expected value of the game $= 2\times\frac{1}{3}+(-2)\times\frac{2}{3}=-\frac{2}{3}$

Mean of a function of a rv

If $Y = h(X)$,

$$\mathrm{E}(Y) = \sum_x h(x)P(X = x) = \sum_x h(x)f(x)\,.$$

Examples: If Z is the number showing on a roll of a "fair" 6-sided die, find $\mathrm{E}(Z^2)$ and $\mathrm{E}\left([Z - 3.5]^2\right)$.

$$\begin{aligned}\mathrm{E}\left(Z^2\right) &= \sum_z z^2 P(Z = z) = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + \cdots \\ &= \frac{1}{6}(1^2 + 2^2 + \cdots + 6^2) = \frac{91}{6} = 15\frac{1}{6}\,.\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left([Z-3.5]^2\right) &= \cdots = \frac{1}{6}\left[(-2.5)^2+(-1.5)^2+(-0.5)^2+0.5^2+1.5^2+2.5^2\right] \\
&= \frac{6.25+2.25+0.25+0.25+2.25+6.25}{6} = \frac{17.5}{6} = 2\frac{5}{6}\,.
\end{aligned}$$

Recall that for this Z, $\mathrm{E}(Z) = 3.5$.

Hence the second example above could be written as $\mathrm{E}\left([Z-\mathrm{E}(Z)]^2\right)$.

This is in fact an important general quantity called the **variance** of Z.

Variance

Variance is defined as the *expected squared difference from the expectation*:

$$\mathrm{Var}(X) \quad = \quad \mathrm{E}\left([X - \mathrm{E}(X)]^2\right) = \mathrm{E}(X^2) - (\mathrm{E}[X])^2$$

We also introduce the **standard deviation** of a rv X as

$$SD(X) = \sqrt{\mathrm{Var}(X)}\,.$$

Variance and SD allow us to compare probability distributions: those with higher variance/SD are *more spread out about the expectation*.

- $\mathrm{E}(aX) = a\,\mathrm{E}(X)$; $\mathrm{E}(X + a) = \mathrm{E}(X) + a$
- $\mathrm{Var}(aX) = a^2\,\mathrm{Var}(X)$; $\mathrm{Var}(X + a) = \mathrm{Var}(X)$

Example: Let X and Y be a random variables with the following p.d.'s:

x	$P(X=x)$
-2	$1/5$
-1	$1/5$
0	$1/5$
1	$1/5$
2	$1/5$

y	$P(Y=y)$
-4	$1/5$
-2	$1/5$
0	$1/5$
2	$1/5$
4	$1/5$

Calculate expected values. Compare variances.

Solution: We have $\mathrm{E}(X)=\mathrm{E}(Y)=0$ and $\mathrm{Var}(X)<\mathrm{Var}(Y)$,

Continuous Random Variables

What to do when there are uncountably infinitely many outcomes, e.g. points on the real line?

In the discrete case, the probability mass function $P(X = x)$ was the main object of interest. In the continuous case this role will be played by the **probability density function**.

The **(cumulative) distribution function** (CDF) of any rv X is

$$F_X(x) = P(X \leq x)\,,$$

viewed as a function of a real variable x.

Area under a curve

We can visualise the distribution in terms of a **probability density function** (PDF) $f_X(x) = \frac{d}{dx}F_X(x)$; then since for any $a < b$,

$$\{X \leq a\} \cup \{a < X \leq b\} = \{X \leq b\}$$

we have

$$\begin{aligned} P(X \leq a) + P(a < X \leq b) &= P(X \leq b) \\ P(a < X \leq b) &= P(X \leq b) - P(X \leq a) \\ &= F_X(b) - F_X(a) = \int_a^b f_X(x)\,dx \end{aligned}$$

Example 10:

(A) Assume that X has the following p.d.f.

$$f_X(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \le x \le 2 \\ 0 & x > 2 \end{cases}$$

The corresponding c.d.f. is given by:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \int_0^x f_X(z)\,dz = \frac{1}{2}\int_0^x z\,dz = \frac{x^2}{4} & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

$$P(0.5 < X < 1.5) = F_X(1.5) - F_X(0.5) = \frac{(1.5)^2}{4} - \frac{(0.5)^2}{4}$$

(B) Assume that X has the following p.d.f.

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x) & x > 0 \\ 0 & x < 0 \end{cases}$$

The corresponding c.d.f. is given by:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \int_0^x f_X(z)\,dz = \lambda \int_0^x \exp(-\lambda z)\,dz = 1 - \exp(-\lambda x) & x > 0 \end{cases}$$

Mean and variance

For a continuous rv X with pdf $f_X(x)$, the expectation is *defined* as

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx\,.$$

Example: Find the expected value in the example (A) above.

Solution: $\mathrm{E}(X) = \int_0^2 x f_X(x)\, dx = \int_0^2 \frac{1}{2}x^2\, dx = \frac{4}{3}$.

Expectation of a function of a cts rv; Variance

In a similar way to the discrete case,

$$\mathrm{E}\left[h(X)\right] = \int_{-\infty}^{\infty} h(x) f_X(x)\, dx\,.$$

In particular, as before,

$$\mathrm{Var}(X) \overset{\text{def}}{=} \mathrm{E}\left(\left[X - \mathrm{E}(X)\right]^2\right) \overset{\text{comp. formula}}{=} \mathrm{E}\left(X^2\right) - \left[\mathrm{E}(X)\right]^2\,.$$

Also as before we define $SD(X) = \sqrt{\mathrm{Var}(X)}$.

Linear function $a + bX$

If we take a linear function of a cts rv, as in the discrete case we get

$$\begin{aligned} \mathrm{E}\,(a + bX) &= a + b\mathrm{E}(X)\,, \\ \mathrm{Var}\,(a + bX) &= b^2\mathrm{Var}(X)\,, \\ \mathit{SD}\,(a + bX) &= |b|\mathit{SD}(X)\,. \end{aligned}$$