

# MAT3320 Assignment 1

Total: 10 marks. Due date: Tuesday, Sept 29, before 6:00pm. In MATH Department (585 King Edward), there is a Drop-Box. You need to put your assignment into the box **before 6:00pm** on the due date. Late assignments will not be accepted.

1. (5 marks) Consider the following differential equation  $y'' + xy' - 3y = 0$ , near  $x_0 = 0$ .

(a) (2 marks) Find the coefficient recursion relation for the general series solution.

(b) (2 marks) Find two linearly independent solutions by solving the recursive relation.

(c) (1 mark) Find the particular solution  $y$  such that  $y(0) = 1$ ,  $y'(0) = 3$ .

**Solution:** (a) Note that  $x_0 = 0$  is an ordinary point. Hence the general solution is given by  $y(x) = \sum_{m=0}^{\infty} a_m x^m$ . Then

$$y'(x) = \sum_{m=0}^{\infty} m a_m x^{m-1},$$
$$y''(x) = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}.$$

Substituting these into DE we have

$$\sum_{m=0}^{\infty} m(m-1) a_m x^{m-2} + x \sum_{m=0}^{\infty} m a_m x^{m-1} - 3 \sum_{m=0}^{\infty} a_m x^m = 0,$$

i.e.,

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{m=0}^{\infty} m a_m x^m - 3 \sum_{m=0}^{\infty} a_m x^m = 0.$$

Collecting like powers of  $x$  we get

$$\sum_{m=0}^{\infty} [(m+2)(m+1) a_{m+2} + a_m(m-3)] x^m = 0.$$

By the property of VAC,

$$(m+2)(m+1) a_{m+2} + (m-3) a_m = 0, \quad m \geq 0; \Rightarrow$$

$$a_{m+2} = -\frac{(m-3) a_m}{(m+2)(m+1)}, \quad m \geq 0.$$

(b) This implies that

$$\begin{aligned} a_2 &= \frac{3}{2}a_0, & a_3 &= \frac{1}{3}a_1, & a_4 &= \frac{1}{4(3)}a_2 = \frac{3(1)}{4(3)2}a_0, \\ a_5 &= 0, a_7 = 0, \dots, a_n = 0 & \text{for all odd } n \geq 5; \\ a_6 &= -\frac{a_4}{6(5)} = -\frac{3(1)}{6!}a_0, & a_8 &= \frac{3(1)3}{8!}a_0; \\ a_{2n} &= 3(-1)^n \frac{(2n-5)!!}{(2n)!}a_0, & n &\geq 3. \end{aligned}$$

Thus we get

$$y = a_1\left(x + \frac{1}{3}x^3\right) + a_0 \left[ 1 + \frac{3}{2}x^2 + \frac{3}{4!}x^4 + 3 \sum_{n=3}^{\infty} (-1)^n \frac{(2n-5)!!}{(2n)!} x^{2n} \right],$$

where

$$y_1 = x + \frac{1}{3}x^3, \quad y_2 = 1 + \frac{3}{2}x^2 + \frac{3}{4!}x^4 + 3 \sum_{n=3}^{\infty} (-1)^n \frac{(2n-5)!!}{(2n)!} x^{2n}.$$

(c) From  $y(0) = 1$ ,  $y'(0) = 3$  we imply that  $a_0 = y(0) = 1$ ,  $a_1 = y'(0) = 3$ . Thus

$$y = 1 + 3x + \frac{3}{2}x^2 + x^3 + \frac{3}{4!}x^4 + 3 \sum_{n=3}^{\infty} (-1)^n \frac{(2n-5)!!}{(2n)!} x^{2n}.$$

2. (2 marks) Legendre's equation  $(1-x^2)y'' - 2xy' = 0$ . Show that  $y(x) = \ln \frac{1+x}{1-x}$  is a solution of the equation.

**Solution:** From  $y(x) = \ln \frac{1+x}{1-x}$  we imply that

$$y'(x) = \frac{1}{1+x} + \frac{1}{1-x}; \quad y''(x) = -\frac{1}{(1+x)^2} + \frac{1}{(1-x)^2}.$$

Substitute them into the differential equation,

$$\begin{aligned} (1-x^2)y'' - 2xy' &= (1-x^2) \left( -\frac{1}{(1+x)^2} + \frac{1}{(1-x)^2} \right) - 2x \left( \frac{1}{1+x} + \frac{1}{1-x} \right) \\ &= -\frac{1-x^2}{(1+x)^2} + \frac{1-x^2}{(1-x)^2} - \frac{2x}{1+x} - \frac{2x}{1-x} \\ &= -\frac{1-x}{1+x} + \frac{1+x}{1-x} - \frac{2x}{1+x} - \frac{2x}{1-x} \\ &= -\frac{1+x}{1+x} + \frac{1-x}{1-x} = -1 + 1 = 0. \end{aligned}$$

3. (1 mark) What is the polynomial solution of the following Legendre's equation:  $(1 - x^2)y'' - 2xy' + 6162y = 0$ ?

**Solution:** From  $n(n+1) = 6162$  we get a positive solution  $n = 78$ . Hence the polynomial solution is  $P_{78}(x)$ .

4. (2 marks) Let  $f(x) = x^3$ ,  $2 < x < 4$ . Find the Fourier-Legendre expansion.

**Solution:**  $P_n(x)$  are only defined on  $-1 < x < 1$ . So we need to make a linear transformation from  $(2,4)$  to  $(-1,1)$ . Let  $s = ax + b$ . Then  $-1 = a(2) + b$ ,  $1 = a(4) + b$ . Then  $a = 1$ ,  $b = -3$ ,  $s = x - 3$ . Let

$$g(s) = f(x) = f(s + 3) = (s + 3)^3 = s^3 + 9s^2 + 27s + 27.$$

Note that

$$s^3 = \frac{2}{5}P_3(s) + \frac{3}{5}s; \quad s^2 = \frac{2}{3}P_2(s) + \frac{1}{3}; \quad s = P_1(s); \quad 1 = P_0(s).$$

We imply that

$$\begin{aligned} g(s) &= \frac{2}{5}P_3(s) + \frac{3}{5}s + 9\left[\frac{2}{3}P_2(s) + \frac{1}{3}\right] + 27s + 27 = \frac{2}{5}P_3(s) + 6P_2(s) + \frac{138}{5}s + 30 \\ &= \frac{2}{5}P_3(s) + 6P_2(s) + \frac{138}{5}P_1(s) + 30P_0(s), \Rightarrow \\ f(x) &= \frac{2}{5}P_3(x - 3) + 6P_2(x - 3) + \frac{138}{5}P_1(x - 3) + 30P_0(x - 3). \end{aligned}$$