# CARLETON UNIVERSITY

**FINAL/DEFERRED EXAMINATION**
April 2007

**DURATION: 3 HOURS**

**Department Name and Course Number:** School of Mathematics and Statistics, MATH1104ABCD.
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AUTHORIZED MEMORANDA

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PLEASE READ THE FOLLOWING INSTRUCTIONS BEFORE PROCEEDING

1. Please count your pages now. **This examination has pages numbered from 1–13**, including this page. The page numbered 13 is blank and is to be used for rough work only. If you feel there is a page missing please report this to your Proctor.
2. **The examination is out of a total of 50 marks** and consists of 32 multiple choice questions with equal weights. **Please fill in only one answer on your Scantron sheets with a pencil** as there is only one answer to any given question. Filling in two or more answers to any question invalidates that question (*i.e.,* you get 0 marks for that question).
3. Please verify that you are in possession of Scantron FORM NO. F-19234-CU (this appears in the lower left of your form). If not, then ask your proctor for a replacement.
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5. ON YOUR SCANTRON FORM: Please fill in your **COURSE NO.** (e.g., MATH1104_) where the "underscore" in the last space refers to your section (e.g., A, B, C, D), in the column labelled "S". Next, write in upper case letters and fill in your **LAST NAME**, **FIRST NAME** and **MIDDLE INITIAL** in the spaces to the left of your Course No. Fill in the **DATE** where appropriate along with your **STUDENT NUMBER** starting with "100..." as required on the Scantron form. Leave the *EXAM VERSION* space blank and fill in your **Gender** where requested.
6. Answer each question by filling in the appropriate box on the Scantron sheet (in pencil) AND by circling the answer on the question sheet.
7. Please make sure your name and student number are filled in accurately on BOTH question paper AND Scantron sheet.
8. Hand in BOTH Scantron sheet AND the exam question paper.
9. Do all rough work on the question paper.
10. Be sure to check the sequence of your answers on the Scantron sheet.

1. Given $A^{-1} = \begin{bmatrix} -1 & 3 \\ 4 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

   What is the solution of the matrix equation $A\mathbf{x} = \mathbf{b}$?

   **(a)** $\begin{bmatrix} 5 \\ 9 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 \\ 9 \end{bmatrix}$ **(c)** $\begin{bmatrix} 5 \\ -7 \end{bmatrix}$ (d) $\begin{bmatrix} -5 \\ 7 \end{bmatrix}$ (e) $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$

2. Suppose that a system of linear equations has the following augmented matrix

   $$\left[\begin{array}{cc|c} 1 & h & 3 \\ 5 & -10 & k \end{array}\right].$$

   If the system is inconsistent, then the values of $h$ and $k$ must be:

   **(a)** $h = -2$, $k = 15$
   **(b)** $h \neq -2$, $k \neq 15$
   **(c)** $h \neq -2$, $k = 15$
   **(d)** $h = -2$, $k \neq 15$
   **(e)** $h = 2$, $k = 15$

3. Find all the values of $h$ such that $\begin{bmatrix} 3 \\ h \\ 7 \end{bmatrix}$ is in Span$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$.

   **(a)** $h = 0$ **(b)** $h \neq 0$ **(c)** $h \neq \pm 1$ **(d)** $h = -1$ **(e)** $h = 1$

4. Let $A$ be an $m \times n$ matrix. Which of the following is true?

   **(a)** The homogeneous equation $Ax = 0$ has the trivial solution if and only if the equation has at least one free variable.

   **(b)** The echelon form of $A$ is unique.

   **(c)** The columns of $A$ are linearly dependent if and only if the only solution of $Ax = 0$ is the trivial one.

   **(d)** If $A$ is the standard matrix of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$, and if $T$ is one-to-one, then the columns of $A$ must span $\mathbb{R}^m$.

   **(e)** The columns of $A$ span $\mathbb{R}^m$ if and only if $A$ has a pivot position in every row.

5. Find all the values of $h$ such that the vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -1 \\ h \end{bmatrix}$ are linearly dependent:

   **(a)** $h = 2$ **(b)** $h \neq 2$ **(c)** $h = -2$ **(d)** $h \neq -2$ **(e)** $h \neq \pm 2$

6. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by $T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x+y-z \\ x+y-w \\ y+z \end{bmatrix}$. Which of the following is true?

   **(a)** $T$ is onto and one-to-one
   **(b)** $T$ is onto, but not one-to-one
   **(c)** $T$ is one-to-one, but not onto
   **(d)** $T$ is neither onto nor one-to-one
   **(e)** none of the above statements is true

7. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that

   $T\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $T\left(\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}\right)$.

   **(a)** $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$ **(b)** $\begin{bmatrix} 9 \\ -6 \end{bmatrix}$ **(c)** $\begin{bmatrix} 11 \\ -2 \end{bmatrix}$ **(d)** $\begin{bmatrix} -11 \\ 2 \end{bmatrix}$ **(e)** $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$

8. For any $n \times n$ matrices $A$, $B$ and $C$, which of the following is always correct?

   (i) If $AB = AC$, then $B = C$.
   (ii) $AB = BA$.
   (iii) $(AB)^T = B^T A^T$.

   **(a)** (i) and (ii) **(b)** (i) and (iii) **(c)** (ii) only **(d)** (iii) only **(e)** (i) only

9. Let $A$ be an $n \times n$ matrix. If $A$ is not invertible, which of the following is correct?

   (i) $A$ has $n$ pivot positions.

   (ii) $A\mathbf{x} = \mathbf{0}$ has at least one non-trivial solution.

   (iii) $A$ is not row equivalent to the identity matrix $I_n$.

   **(a)** (i) and (ii) **(b)** (i) and (iii) **(c)** (ii) and (iii) **(d)** (i) only **(e)** (ii) only

10. Let $B = \begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix}$ and $(A^T + B)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. What is the matrix $A$?

    **(a)** $\begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 6 & 4 \\ -1 & 4 \end{bmatrix}$ **(c)** $\begin{bmatrix} 4 & 0 \\ 4 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} -4 & 0 \\ -4 & -1 \end{bmatrix}$ **(e)** $\begin{bmatrix} -4 & -1 \\ -4 & 0 \end{bmatrix}$

11. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$. If $A^{-1} = B = [b_{ij}]$, what is $b_{12}$?

    **(a)** 2 **(b)** 1 **(c)** 0 **(d)** $-1$ **(e)** $-2$

12. Let $A$ be a $5 \times 7$ matrix such that $\dim \mathrm{Col} A = 4$. What is $\dim \mathrm{Nul} A$?

    **(a)** 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** 4

13. Let $A$ be a $3{\times}3$ matrix with an eigenvalue $\lambda = 0$. You are given that $A \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

    What is the eigenvector of $A$ corresponding to $\lambda = 0$?

    **(a)** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ **(e)** $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

14. Let $A$ be an invertible $n \times n$ matrix. Which of the following is false?

**(a)** $\text{Col}A = \mathbb{R}^n$.
**(b)** $\text{dimNul}A = n$.
**(c)** $\text{rank}A = n$.
**(d)** $\text{dimNul}A = 0$.
**(e)** $\text{dimCol}A + \text{dimNul}A = n$

15. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 4 & 4 \\ 5 & 5 & 5 & 5 & 6 \end{bmatrix}$. What is $\det A$ ?

**(a)** 0 **(b)** 6 **(c)** 12 **(d)** 24 **(e)** 48

16. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $B = \begin{bmatrix} g & h & i \\ -2d & -2e & -2f \\ a & b & c \end{bmatrix}$.

If $\det(A) = 2$, what is $\det(B)$?

**(a)** 2 **(b)** 4 **(c)** $-4$ **(d)** $-2$ **(e)** 0

17. Let $A$ be an $2 \times 2$ matrix and $\det A = -3$. What is $\det(2A^2A^TA^{-1})$ ?

**(a)** $-36$ **(b)** $-18$ **(c)** 18 **(d)** 36 **(e)** 9

18. Consider the matrix equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 & -k \\ 0 & 1 & 1 \\ k & 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } \det(A) = 1 + k^2.$$

What is the value of $x_1$? (Hint: Use Cramer's Rule)

**(a)** $\frac{k}{1+k^2}$ **(b)** $\frac{k^2}{1+k^2}$ **(c)** $1 + k^2$ **(d)** $k$ **(e)** 0

19. What is the area of the parallelogram whose vertices are: $(0,0)$, $(2,3)$, $(-2,4)$ and $(0,7)$?

    **(a)** 6 **(b)** 7 **(c)** 8 **(d)** 12 **(e)** 14

20. Let $z = \cos(\pi/16) + i\sin(\pi/16)$. What is $z^4$ in the standard form?

    **(a)** $\frac{1}{\sqrt{2}}(1+i)$ **(b)** $\frac{1}{\sqrt{2}}(1-i)$ **(c)** $1+i$ **(d)** $1-i$ **(e)** 1

21. What is the standard form of the complex number $\dfrac{2+4i}{1+i}$ ?

    **(a)** $1-3i$ **(b)** $1-i$ **(c)** $1+2i$ **(d)** $3+i$ **(e)** $3-i$

22. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{bmatrix}$. What are the eigenvalues of $A$?

    **(a)** $1, \ -2, \ 2$ **(b)** $1, \ 2, \ -4$ **(c)** $2, \ 2, \ 4$ **(d)** $0, \ 1, \ 2$ **(e)** $-2, \ 1, \ 6$

23. Let $A = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$. You are given that $\lambda = 1$ is an eigenvalue of $A$.

    What is the dimension of the corresponding eigenspace?

    **(a)** 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** 4

24. Let $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

    What is the maximum number of linearly independent eigenvectors of I?

    **(a)** 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** $\infty$

25. Which of the following sets are subspaces of $R^3$.

$$U = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \middle| \; a+b=1 \right\}, \quad V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \; a-b=c \right\},$$

$$W = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad H = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

**(a)** $U$ and $V$ **(b)** $V$ and $W$ **(c)** $V$ and $H$ **(d)** $U$ and $H$ **(e)** $W$ and $H$

26. Let $A = \begin{bmatrix} 1 & -1 \\ 9 & 1 \end{bmatrix}$. What are the eigenvalues of $A$?

**(a)** $1 \pm 3i$ **(b)** $-1 \pm 3i$ **(c)** $\pm 3i$ **(d)** $\pm 1$ **(e)** 1 and $3i$

27. Let $A$ be a $2 \times 2$ matrix with eigenvectors $v_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

corresponding to eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, respectively. What is the matrix $A^{99}$?

**(a)** $\begin{bmatrix} 7 & -2 \\ -7 & 24 \end{bmatrix}$ **(b)** $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 7 & -2 \\ 24 & -7 \end{bmatrix}$ **(d)** $\begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}$ **(e)** $\begin{bmatrix} 7 & 24 \\ -2 & -7 \end{bmatrix}$

28. Let $x = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

What is the orthogonal projection of the vector $x$ onto the vector $u$?

**(a)** $\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$ **(d)** $\begin{bmatrix} 4 \\ -4 \\ -7 \end{bmatrix}$ **(e)** $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

29. Let $u_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, and $u_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$.
You are given that $S = \{u_1,\ u_2,\ u_3\}$ is an orthogonal basis for $R^3$.

Let $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. What is the value of $c_1$ in the equation $x = c_1u_1 + c_2u_2 + c_3u_3$?

**(a)** $-1$ **(b)** $1$ **(c)** $-2$ **(d)** $2$ **(e)** $4$

30. Let $x = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

What is the distance from $x$ to the line through $u$ and the origin?

(Hint: The projection of the vector $x$ onto the vector $u$ is $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$.)

**(a)** $\sqrt{20}$ **(b)** $\sqrt{45}$ **(c)** $\sqrt{50}$ **(d)** $\sqrt{90}$ **(e)** $\sqrt{125}$

31. Let $u = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. What is the angle between the vectors $u$ and $v$?

**(a)** $\frac{\pi}{3}$ **(b)** $\frac{-\pi}{3}$ **(c)** $\frac{2\pi}{3}$ **(d)** $0$ **(e)** $\frac{\pi}{2}$

32. Let $A$ be a $3 \times 3$ matrix with the eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$.

You are given that $A - I \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A - 2I \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

If $A = PDP^{-1}$ with $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, what is the matrix $P$?

**(a)** $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 0 & 1 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **(e)** $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

## ANSWERS

| | | | |
|---|---|---|---|
| 1. (d) | 9. (c) | 17. (d) | 25. (c) |
| 2. (d) | 10. (e) | 18. (a) | 26. (a) |
| 3. (a) | 11. (e) | 19. (e) | 27. (e) |
| 4. (e) | 12. (d) | 20. (a) | 28. (a) |
| 5. (c) | 13. (e) | 21. (d) | 29. (b) |
| 6. (b) | 14. (b) | 22. (e) | 30. (b) |
| 7. (b) | 15. (d) | 23. (c) | 31. (c) |
| 8. (d) | 16. (b) | 24. (d) | 32. (e) |