MATH 1005H Fall 2015

Test One Solutions

1. (6 marks) Solve the initial-value problem

$$2y' = \frac{3x^2 - \sin(x)}{y}, \quad y(0) = -3.$$

Solution: This is a separable equation. Rearranging, we have

$$2yy' = 3x^2 - \sin(x).$$

Integrating with respect to x (and using the substitution rule on the left) yields

$$\int 2y dy = \int (3x^2 - \sin(x)) dx$$
$$y^2 = x^3 + \cos(x) + c.$$

Thus $y = \pm\sqrt{x^3 + \cos(x) + c}$ is the general solution. Since $y(0) = -3$, we have

$$-3 = \pm\sqrt{1 + c}.$$

Thus we must choose the negative sign and set $1 + c = 9$, and so $c = 8$. The solution to the initial value problem is

$$y = -\sqrt{x^3 + \cos(x) + 8}.$$

2. (6 marks) Find the general solution of the equation

$$xy' = y\big(1 + \ln(y) - \ln(x)\big), \quad x > 0,\ y > 0.$$

Solution: Rearranging, we have

$$y' = \frac{y}{x}\left(1 + \ln\left(\frac{y}{x}\right)\right).$$

This is a homogeneous equation. We set $u = \frac{y}{x}$, so that $y = ux$ and $y' = u + xu'$. Then

$$u + xu' = u(1 + \ln(u)).$$

Rearranging, we have

$$\frac{1}{u\ln(u)} u' = \frac{1}{x}.$$

This is a separable equation. Integrating both sides with respect to x (and using the substitution rule on the left), we have

$$\int \frac{1}{u\ln(u)} du = \int \frac{1}{x} dx$$
$$\ln|\ln(u)| = \ln|x| + c_1$$
$$|\ln(u)| = e^{c_1}|x|$$
$$\ln(u) = \pm e^{c_1} x$$
$$\ln(u) = cx$$
$$u = e^{cx}$$

(to evaluate the integral on the left, the substitution $t = \ln(u)$ may be made). Thus

$$y = xu = xe^{cx}$$

is the general solution.

3. (6 marks) Find the general solution of the equation

$$x^2y' + 2x^2y = x^6 + x^3.$$

Solution: This is a linear equation. We divide by x^2 to put it in standard form:

$$y' + 2y = x^4 + x.$$

We choose the integrating factor

$$I(x) = e^{\int 2dx} = e^{2x}.$$

Multiplying the equation by e^{2x} yields

$$\begin{aligned} e^{2x}y' + 2e^{2x}y &= e^{2x}x^4 + e^{2x}x \\ (e^{2x}y)' &= e^{2x}x^4 + e^{2x}x. \end{aligned}$$

We integrate to get

$$e^{2x}y = \int \left(e^{2x}x^4 + e^{2x}x\right) dx = \int e^{2x}\left(x^4 + x\right) dx.$$

The integral can be evaluated by using the table method or by applying integration by parts repeatedly (for what it's worth, there was a typo in the question. I had intended this integral to be easier to evaluate!). Doing this, we have

$$e^{2x}y = e^{2x}\left(\frac{1}{2}x^4 - x^3 + \frac{3}{2}x^2 - x + \frac{1}{2}\right) + c.$$

Thus the general solution is

$$y = \frac{1}{2}x^4 - x^3 + \frac{3}{2}x^2 - x + \frac{1}{2} + ce^{-2x}.$$

4. (6 marks) Find the general solution of

$$2xe^y + (x^2e^y + 4y^3)y' = 0.$$

Solution: We have $P_y = 2xe^y$ and $Q_x = 2xe^y$. The equation is exact because P_y and Q_x are continuous in the entire plane (which is simply connected), and $P_y = Q_x$. A potential function f exists. We use the fact that $f_x = P$ and integrate with respect to x to get

$$f(x, y) = x^2e^y + g(y).$$

We now use the fact that we must have $f_y = Q$ to get

$$x^2e^y + g'(y) = x^2e^y + 4y^3.$$

Thus $g'(y) = 4y^3$ and so $g(y) = y^4 + c_1$. A potential function is

$$f(x, y) = x^2e^y + y^4 + c_1.$$

To get the solution of the equation, we set $f(x, y) = c_2$. Combining constants, this yields the solution

$$x^2e^y + y^4 = k.$$

5. (6 marks) Find the general solution of the equation

$$y' + y = -2e^{2x}y^2.$$

Solution: This is a Bernoulli equation with $\alpha = 2$. We set $u = y^{1-\alpha} = y^{-1}$. Then $y = u^{-1}$, and so $y^2 = u^{-2}$ and $y' = -u^{-2}u'$. Substituting this in to the equation, we get

$$-u^{-2}u' + u^{-1} = -2e^{2x}u^{-2}.$$

Multiplying by $-u^2$ yields

$$u' - u = 2e^{2x}.$$

This is a linear equation. We choose the integrating factor

$$I(x) = e^{\int -1dx} = e^{-x}.$$

Multiplying the equation by e^{-x} yields

$$\begin{aligned} e^{-x}u' - e^{-x}u &= 2e^x \\ (e^{-x}u)' &= 2e^x. \end{aligned}$$

We integrate to get

$$\begin{aligned} e^{-x}u &= 2e^x + c \\ u &= 2e^{2x} + ce^x. \end{aligned}$$

Thus the general solution is

$$y = u^{-1} = \frac{1}{2e^{2x} + ce^x}.$$