Geometry of the Complex Numbers (section 1.3)

- Recall the "complex plane": each complex number $z = a + bi$ can be represented as a point (a, b) in the complex plane, where the horizontal axis is called the "real axis" and the vertical axis is called the "imaginary axis".

- **Remarks:**

1) Addition of to complex numbers = Addition of two vectors in the complex plane.

2) Negation of a complex number = negation of the corresponding vector.

3) Multiplication by real numbers = Scalar multiplication

4) Complex conjugate of a complex number = Reflection of the vector through the real axis.

$z = a + bi$, then $\bar{z} = a - bi$

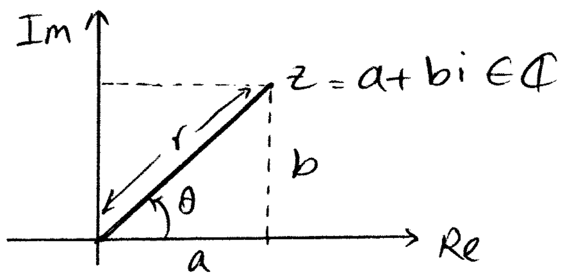
5) Modulus of a complex number = length of the corresponding vector

$z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.

polar form of complex numbers (section 1.4)

"polar coordinates" : "$z = re^{i\theta}$"

If $z = a + bi$ with $a, b \in \mathbb{R}$, then



$$\frac{a}{r} = \cos(\theta)$$

$$\frac{b}{r} = \sin(\theta)$$

$$r = |z| = \sqrt{a^2 + b^2}$$

Therefore, $z = a + bi = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta))$.

<u>Note</u>: θ is not uniquely determined, since $\theta' = \theta + 2n\pi$, $n \in \mathbb{Z}$, also works. θ is called the "argument of z" and is denoted by "$\theta = \arg(z)$".

If we assume in addition that $-\pi < \theta \leq \pi$, then θ is called the "<u>principal</u> argument of z" and is denoted by "$\theta = \mathrm{Arg}(z)$".

• <u>Euler (1748)</u>: $e^{z} = 1 + z + \frac{z^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

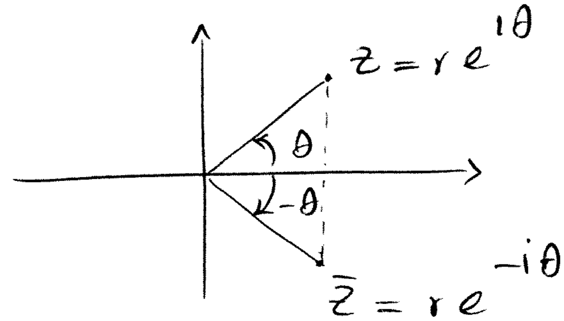
Hence, by comparing power series of trig functions, we get

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

or, the polar form of complex numbers: $\boxed{z = re^{i\theta}}$

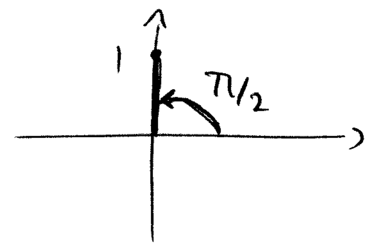
Remarks :

- $re^{i\theta} = r'e^{i\theta'} \Leftrightarrow r = r', \ \theta - \theta' = 2n\pi, \ n \in \mathbb{Z}.$
- $\overline{(re^{i\theta})} = re^{-i\theta},$
- $|e^{i\theta}| = 1$, for any θ.



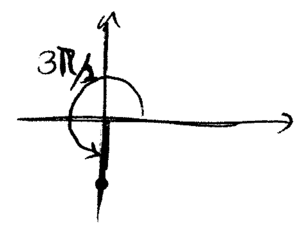
Examples :

1) $i = 1 \cdot e^{i\pi/2} = 1 \cdot (\cos(\pi/2) + i\sin(\pi/2))$



$|i| = \sqrt{0^2 + 1^2} = 1$

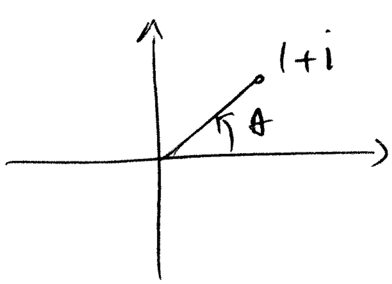
2) $-i = 1 \cdot e^{i\frac{3\pi}{2}} = 1 \cdot (\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2}))$



$|-i| = \sqrt{0^2 + (-1)^2} = 1$

3) $1 + i = \frac{\sqrt{2}}{2} e^{i\pi/4}$, $|1+i| = \sqrt{1+1} = \sqrt{2} = r$

$$\left.\begin{aligned} \cos(\theta) &= \frac{1}{\sqrt{2}} \\ \sin(\theta) &= \frac{1}{\sqrt{2}} \end{aligned}\right\} \Rightarrow \theta = \pi/4$$



Multiplying complex numbers in polar form (section 1.5)

$z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2)(e^{i\theta_1} \cdot e^{i\theta_2})$$

$$= (r_1 r_2)(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= (r_1 r_2)\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\right]$$

$$= r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right] = r_1 r_2 e^{i(\theta_1+\theta_2)} \; !$$

Similarly, if $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)} \; !$$

Example: compute polar form of $\dfrac{-\sqrt{3}+i}{1+i}$:

$$\left.\begin{array}{l} z_1 = -\sqrt{3}+i \Rightarrow |z_1| = r_1 = |-\sqrt{3}+i| = \sqrt{3+1} = 2 \\ \left.\begin{array}{l} \cos\theta_1 = -\dfrac{\sqrt{3}}{2} \\ \sin\theta_1 = \dfrac{1}{2} \end{array}\right\} \Rightarrow \theta_1 = 5\pi/6 \end{array}\right\} \Rightarrow z_1 = 2e^{i5\pi/6}$$

$z_2 = 1+i \Rightarrow |z_2| = r_2 = \sqrt{2}$ and $\theta_2 = \pi/4$ (from previous example

$$\Rightarrow \frac{-\sqrt{3}+i}{1+i} = \frac{2e^{i5\pi/6}}{\sqrt{2}\,e^{i\pi/4}} = \sqrt{2}\,e^{i\pi(5/6-\frac{1}{4})} = \sqrt{2}\,e^{i\frac{7\pi}{12}} \checkmark$$

The fundamental theorem of Algebra (section 1.6)

Theorem (1.1):

Every polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ with complex coefficients ($a_n, a_{n-1}, \ldots, a_0 \in \mathbb{C}$) factors completely into linear factors of the form $c_i x + d_i$, with $c_i, d_i \in \mathbb{C}$, i.e.

$$p(x) = (c_1 x + d_1)(c_2 x + d_2) \cdots (c_n x + d_n)$$

Example: $x^2 + x + 5 = p(x)$

Discriminant $D = b^2 - 4ac = 1^2 - 4.1.5 = -19$.

So, this polynomial has no real roots:

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-19}}{2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-19}$$

$$= -\frac{1}{2} \pm \frac{i}{2}\sqrt{19}$$

(Complex roots).

Remark:

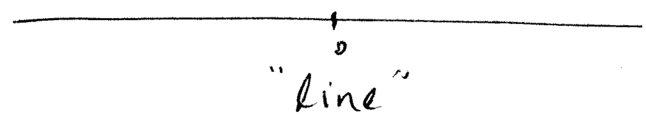
The fundamental theorem of Algebra doesn't say anything about "finding" the roots. It just says that they exist.

CHAPTER TWO : "Vector Geometry" (Review)

2.1 Vectors in $\mathbb{R}^n$:

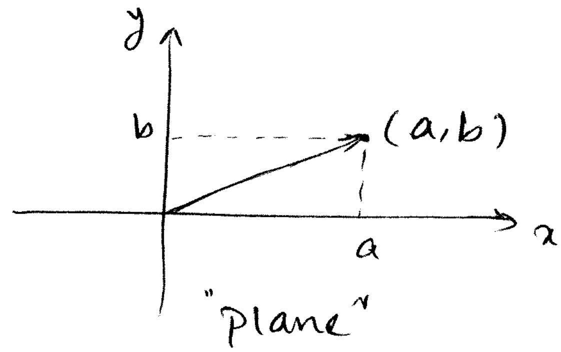
Algebra | **Geometry**

- $\mathbb{R}$: set of real numbers (scalars)



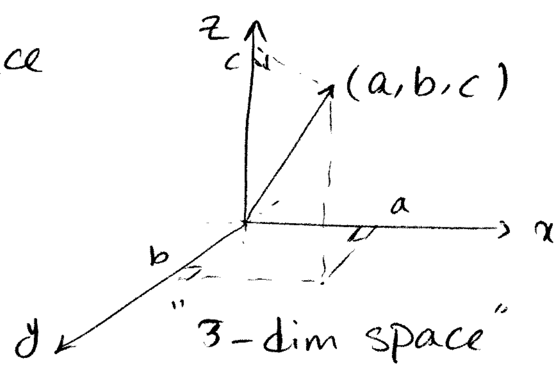
- $\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$.

  $(x,y) = \vec{u}$: vectors in 2-space



- $\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$

  $(x,y,z) = \vec{v}$: vectors in 3-space



⋮ no need to stop here !

- $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$

  $\vec{x} = (x_1, x_2, \ldots, x_n)$ are vectors in "<u>n-space</u>"

  $(n \in \mathbb{Z}, n > 0)$ (hard to visualize them).

2.2 Manipulation of vectors in $\mathbb{R}^n$:

(1) Equality : $(x_1, \cdots x_n) = (x'_1, \cdots, x'_n) \Leftrightarrow x_i = x'_i$ for all $i \in \{1, \cdots, n\}$.

(2) Addition : $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$

$\vec{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$

$$\vec{x} + \vec{y} = (x_1 + y_1, \cdots, x_n + y_n) \in \mathbb{R}^n$$

(3) Zero vector : $\vec{0} = (0, 0, \cdots, 0) \in \mathbb{R}^n$

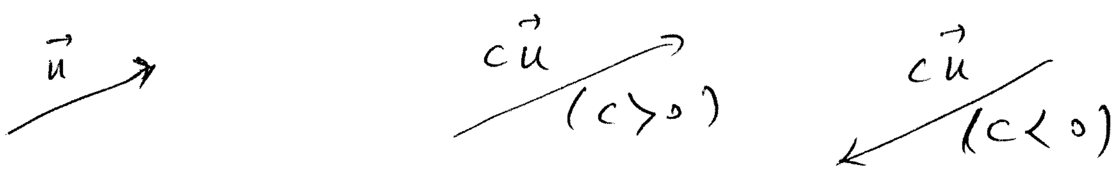
(4) Negative : if $\vec{x} = (x_1, \cdots, x_n)$, then $-\vec{x} = (-x_1, \cdots, -x_n) \in \mathbb{R}^n$

(5) Multiplication by <u>scalars</u> :

let $a \in \mathbb{R}$ be a scalar, then

$$c \cdot \vec{x} = c(x_1, \cdots, x_n) = (cx_1, cx_2, \cdots, cx_n) \in \mathbb{R}^n$$

example: (in $\mathbb{R}^2$)



Geometric Interpretations:

- Equality : two vectors have the same magnitude & direction.
- Addition : Parallelogram rule, or head-to-tail rule.
- Zero vector: a vector with no length.
- Negative : the same arrow in opposite direction.
- Scalar Multiplication: scale the vector by $|c|$, and change of direction if $c<0$.

2.3 Linear Combinations:

Definition 2.1:

If $k_1, k_2, \cdots, k_m \in \mathbb{R}$ are scalars, and $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m \in \mathbb{R}^n$ are vectors, then

$$k_1 \vec{u}_1 + k_2 \vec{u}_2 + \cdots + k_m \vec{u}_m$$

is called a "linear combination" of $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m$.

example: $k_1 = 2, \; k_2 = \sqrt{5}$

$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, then

$$k_1 \vec{u}_1 + k_2 \vec{u}_2 = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+2\sqrt{5} \\ -\sqrt{5} \end{pmatrix} \in \mathbb{R}^2$$

2.4 properties of vector manipulations:

All usual properties hold: suppose $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ & $c, c' \in \mathbb{R}$, then

- $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
- $\vec{x} + \vec{0} = \vec{x}$
- $\vec{x} + (-\vec{x}) = \vec{0}$
- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
- $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
- $(c + c')\vec{x} = c\vec{x} + c'\vec{x}$
- $(cc')\vec{x} = c(c'\vec{x})$
- $1\vec{x} = \vec{x}$

2.5 The Inner (Dot) Product in $\mathbb{R}^n$:

$n = 3$: Recall that if $\vec{u} = (x_1, x_2, x_3)$ and $\vec{v} = (y_1, y_2, y_3)$, then

$$\vec{u} \cdot \vec{v} = x_1 y_1 + x_2 y_2 + x_3 y_3 \quad (\text{scalar})$$

With the inner (dot) product, we can define :

(1) $\|\vec{u}\| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{\vec{u} \cdot \vec{u}}$: the "length" or "norm" of $\vec{u}$.

(2) $\|\vec{u} - \vec{v}\|$ is the distance between $\vec{u}$ & $\vec{v}$. (see below)

Note :
- $\|\vec{u}\|$ is a real number ≥ 0.
- $\|\vec{u}\| = 0 \Leftrightarrow \vec{u} = \vec{0}$
- $\|a\vec{u}\| = |a| \cdot \|\vec{u}\|$, where $|a|$ is the absolute value of a. $(a \in \mathbb{R})$

Examples : $\vec{u} = (1, -3, 4) \Rightarrow \|\vec{u}\| = \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{26}$

$-3\vec{u} = (-3, 9, -12) \Rightarrow \|-3\vec{u}\| = |-3| \cdot \|\vec{u}\| = 3\sqrt{26} = \sqrt{234}$

also : $\|-3\vec{u}\| = \sqrt{(-3)^2 + 9^2 + (-12)^2} = \sqrt{234}$.

Note :
- $\|\vec{u} - \vec{v}\| = \|\vec{v} - \vec{u}\|$, $\vec{u} = (x_1, x_2, x_3), \vec{v} = (y_1, y_2, y_3)$
- $\|\vec{u} - \vec{v}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$.

Example : $\vec{u} = (1, 0, 2), \vec{v} = (2, -2, 1) \Rightarrow \vec{u} - \vec{v} = (-1, 2, 1)$

$\Rightarrow \|\vec{u} - \vec{v}\| = \sqrt{(-1)^2 + 2^2 + 1^2} = \sqrt{6}$.

Definition (2.2): Let $\vec{u}=(x_1,\dots,x_n)$ & $\vec{v}=(y_1,\dots,y_n)$ be vectors in $\mathbb{R}^n$. Then their "dot Product" is defined to be

$$\vec{u}\cdot\vec{v} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

and the "norm" of $\vec{u}$ is defined to be

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{x_1^2+\cdots+x_n^2}$$

- $\mathbb{R}^n$ (equipped with the dot product) is also called a "Euclidean space".

Example: $\vec{u}=(1,2,-1,0,1)$, $\vec{v}=(1,3,2,1,1)$

$$\vec{u}\cdot\vec{v} = 1+6-2+0+1 = 6$$

$$\|\vec{u}\| = \sqrt{1^2+2^2+(-1)^2+0+1^2} = \sqrt{7}$$

$$\|\vec{v}\| = \sqrt{1+9+4+1+1} = \sqrt{16} = 4$$

Note:
- For any vector $\vec{u}\in\mathbb{R}^n$: $\|\vec{u}\|=0 \iff \vec{u}=\vec{0}$
- $\|\vec{u}\|$ is a real number ≥ 0.
- $\|a\vec{u}\| = |a|\,\|\vec{u}\|$, where $|a|$ is the absolute value of $a\in\mathbb{R}$.

2.6 Orthogonality:

Recall that in $\mathbb{R}^2$ & $\mathbb{R}^3$: $\vec{u}\cdot\vec{v}=0 \iff \vec{u}$ & $\vec{v}$ are "orthogonal" (perpendicular).

Definition (2.3): Let $\vec{u},\vec{v}\in\mathbb{R}^n$. Then $\vec{u}$ & $\vec{v}$ are said to be "orthogonal" if

$$\boxed{\vec{u}\cdot\vec{v}=0}$$

Example: $\vec{u}=(1,2,-2,1)$ & $\vec{v}=(4,1,3,0)$ of $\mathbb{R}^4$ are orthogonal, since $\vec{u}\cdot\vec{v}=0$

2.7 Angles between vectors in $\mathbb{R}^n$:

Recall that we can determine the angle between vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. Now, the question is: can that be generalized to $\mathbb{R}^n$?

Theorem 2.1: (Cauchy-Schwarz Inequality) For $\vec{u}, \vec{v} \in \mathbb{R}^n$, we have

$$\boxed{|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\cdot\|\vec{v}\|} \qquad (\text{Remember } \vec{u}\cdot\vec{v} \in \mathbb{R})$$

Applying theorem (2.1) to simplify $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}\cdot\vec{v})$ yields

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

"triangle inequality"

Definition 2.4: If $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $\vec{u}, \vec{v} \neq \vec{0}$, then the angle θ between $\vec{u}$ & $\vec{v}$ is defined to be the number θ satisfying:

(1) $\cos(\theta) = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$, (2) $0 \leq \theta \leq \pi$.

Example: $\vec{u} = (0,0,3,4,5)$, $\vec{v} = (-1,1,-1,1,2)$, $\vec{u}$ & $\vec{v} \in \mathbb{R}^5$, then

$$\cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \frac{11}{\sqrt{50}\,\sqrt{8}} = \frac{11}{20}$$

$$\Rightarrow \theta = \arccos\left(\frac{11}{20}\right).$$

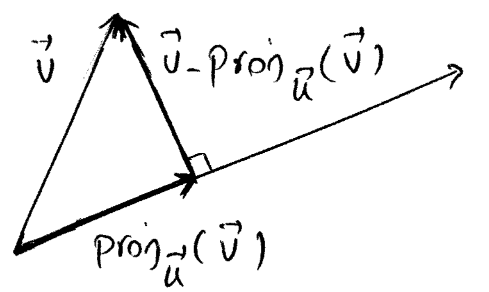
Note: Two vectors $\vec{u}$ and $\vec{v}$ are parallel if the angle between them is either 0° or π.

2.8 Orthogonal Projections onto Lines in $\mathbb{R}^n$:

$\vec{u}$ & $\vec{v}$ are non-zero vectors, then the "projection of $\vec{v}$ onto $\vec{u}$", which is denoted by "$\text{proj}_{\vec{u}}(\vec{v})$", is the unique vector satisfying:

(1) $\text{proj}_{\vec{u}}(\vec{v})$ is parallel to $\vec{u}$ (a scalar multiple of $\vec{u}$).

(2) $\vec{v} - \text{proj}_{\vec{u}}(\vec{v})$ is orthogonal to $\vec{u}$ (their dot product is zero).



$$\vec{v} = (\text{proj}_{\vec{u}}(\vec{v})) + (\vec{v} - \text{proj}_{\vec{u}}(\vec{v}))$$

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\,\vec{u}$$ (scalar multiple of $\vec{u}$)

Example: $\vec{u} = (3,1,-1)$, $\vec{v} = (2,0,+2)$

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{(2,0,2)\cdot(3,1,-1)}{\left(\sqrt{3^2+1^2+(-1)^2}\right)^2}(3,1,-1)$$

$$= \frac{6+0-2}{9+1+1}(3,1,-1) = \frac{4}{11}(3,1,-1)$$

$$\vec{v} - \text{proj}_{\vec{u}}(\vec{v}) = (2,0,2) - \frac{4}{11}(3,1,-1)$$

Check: $\vec{u}\cdot(\vec{v} - \text{proj}_{\vec{u}}(\vec{v})) = 0$, so $\vec{u}$ and $\vec{v} - \text{proj}_{\vec{u}}(\vec{v})$ are orthogonal, as required.