Chapter 1 : Complex Numbers

Hierarchy of number systems :

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ Natural numbers — Cannot solve $x+5=1$

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ Integers — cannot solve $2x=5$

$\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$ Rationals — cannot solve $x^2 = 2$

$\mathbb{R} = \{$ rationals and irrationals $\}$ Real numbers — cannot solve $x^2=-1$
(like $\sqrt{2}, e, \pi, \dots$)

$\mathbb{C} = \{$ complex numbers $\}$ can solve all of the above!

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Denote one solution of $x^2+1=0$ by "i" (stands for "imaginary")
(Euler)

$$\boxed{i^2 := -1} \quad \text{or} \quad \boxed{i := \sqrt{-1}}$$

For any negative real number "a", define:

$$\boxed{\sqrt{a} := \left(\sqrt{|a|}\right) i}$$

Definition: The set of "complex numbers"

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

denote $z = a + bi$, then

$$z = a + bi \in \mathbb{C}, \qquad a, b \in \mathbb{R}$$

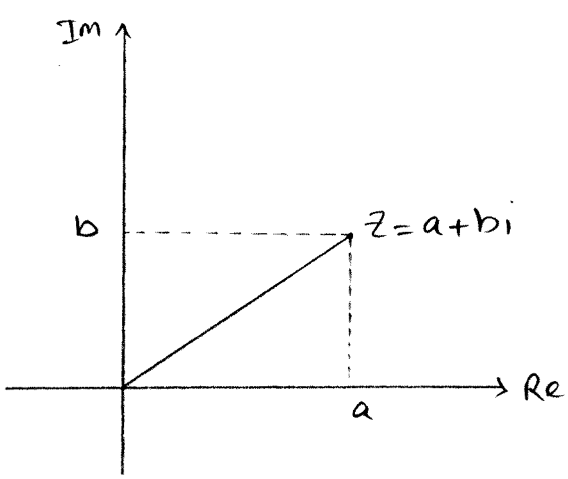
where

$$\begin{cases} a = \mathrm{Re}(z) = \underline{\text{real}} \text{ part of } z \\ b = \mathrm{Im}(z) = \underline{\text{imaginary}} \text{ part of } z \end{cases}$$

Example: $b = 0$, then $z = a \in \mathbb{R}$ (a is a real number)

$a = 0$, then $z = bi \in \mathbb{C}$ (called purely imaginary)

Remark: Real numbers form the real number line (or axis);
Complex numbers form the complex "plane":



$$\underline{i^2 = -1}$$

Algebra of Complex Numbers (Section 1.2)

- Equality: $a+bi = c+di \iff a=c$ and $b=d$;
- Addition: $(a+bi)+(c+di) = (a+c)+(b+d)i$
- Multiplication: $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$

(Exercise)

- subtraction: $(a+bi)-(c+di) = (a-c)+(b-d)i$
- Negatives: $-(a+bi) = -a-bi$

Remark: $\mathbb{C}$ satisfies all the usual arithmetical properties of $\mathbb{R}$, except $\mathbb{C}$ has no "good" order, e.g. "$z>0$" makes no sense in general

- Devision: (via "rationalizing the denominator")

First, note $z\cdot\bar{z} = (a+bi)(a-bi) = a^2 - abi + abi + (bi)(-bi)$

$$= a^2 - b^2 i^2 = a^2+b^2 = |z|^2$$

(Here, $\bar{z} = a-bi$ denotes the "complex conjugate" of $z = a+bi$, and $|z| = \sqrt{a^2+b^2}$ denotes the "absolute value", or "modulus", of z)

Now, consider two complex numbers:

$z = a+bi \in \mathbb{C}$

$w = c+di \in \mathbb{C}$

and the question is to find $\frac{z}{w}$.

$$\frac{z}{\omega} = \frac{z}{\omega}\cdot 1 = \frac{z}{\omega}\cdot\frac{\bar{\omega}}{\bar{\omega}} = \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}$$

$$= \frac{(a+bi)(c-di)}{c^2+d^2} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\,i \qquad \text{(a complex number!)}$$

Example: $z = 2+3i$, $\omega = -3-i$

$$\frac{z}{\omega} = \frac{2+3i}{-3-i}\cdot\frac{-3+i}{-3+i} = \frac{(2+3i)(-3+i)}{(-3)^2+(-1)^2}$$

$$= \frac{-6+2i-9i+3i^2}{10} = \frac{-9-7i}{10} = -\frac{9}{10} - \frac{7}{10}\,i$$

Some Other Properties of Complex Numbers: suppose $z, \omega \in \mathbb{C}$, $a \in \mathbb{R}$

1) $\frac{1}{z} = \frac{1}{z}\cdot\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2}$

2) $\overline{z+\omega} = \bar{z}+\bar{\omega}$

3) $\overline{az} = a\bar{z}$

4) $\overline{z\omega} = \bar{z}\,\bar{\omega}$

5) $\overline{\left(\frac{z}{\omega}\right)} = \frac{\bar{z}}{\bar{\omega}}$

6) $\overline{(\bar{z})} = z$

7) $\bar{z} = z$ iff $z \in \mathbb{R}$

8) $\bar{z} = -z$ iff "z" is purely imaginary

9) $|z| \in \mathbb{R}$ & $|z| \geq 0$

10) $|z| = |\bar{z}|$

11) $|z\omega| = |z|\cdot|\omega|$

12) $\left|\frac{z}{\omega}\right| = \frac{|z|}{|\omega|}$

13) $|z+\omega| \leq |z| + |\omega|$

("Triangle inequality")