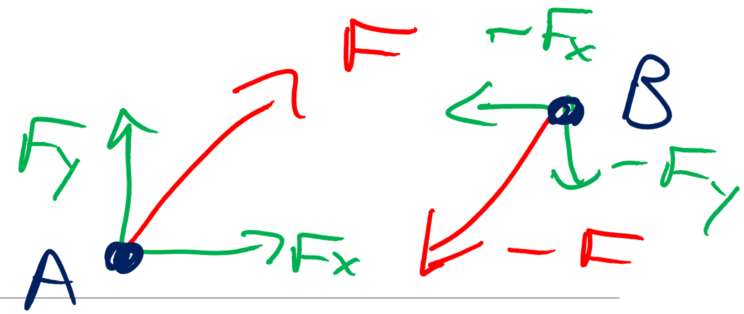
GNG 1105E – Engineering Mechanics

CHAPTER 3 – RIGID BODIES (CONTINUED)

3.12 Moment of a couple



Two forces **F** and **–F** having the same magnitude, parallel lines of action, and opposite sense are said to form a couple

It is clear that the sum of the components of the two forces in any direction is zero

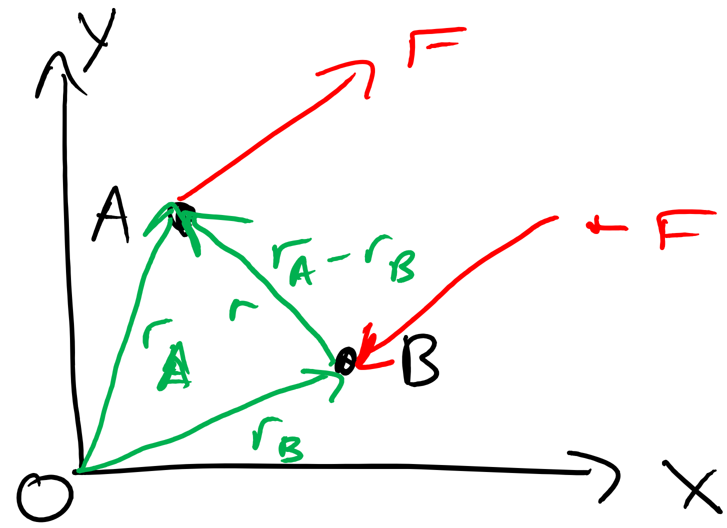
However, the sum of the moments of the two forces about a given point is not zero

These forces tend to rotate the body and not translate it

3.12 Moment of a couple

If $\mathbf{r}_A$ & $\mathbf{r}_B$ are the position vectors of the points of application of **F** and **–F**, then the sum of the moments of the two forces about O is:

$$\mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}$$



3.12 Moment of a couple

Setting $\mathbf{r}_A - \mathbf{r}_B = \mathbf{r}$, where **r** is the vector joining the points of application of the two forces, the sum of the moments of **F** & **-F** about O is:

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

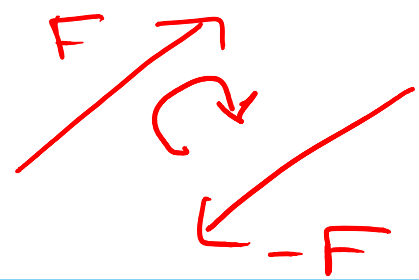
3.12 Moment of a couple

The vector **M** is called the moment of the couple; it is a vector perpendicular to the plane of the two forces and its magnitude is:

$$M = rF\sin\theta = Fd$$

Where d is the perpendicular distance between the lines of action of **F** & **-F**

The sense of **M** is defined by the right-hand rule



3.12 Moment of a couple

Two couples will have equal moments if:

$$F_1 d_1 = F_2 d_2$$

And if they lie in the same plane or in parallel planes and have the same sense

3.13 Equivalent couples

Read Section 3.13 in the textbook (p. 127-129) for proof that two couples having the same moment and acting in the same or parallel planes are equivalent and will produce the same effect on a rigid body

Keep in mind that the only motion a couple can impart to a rigid body is rotation (no translation!)

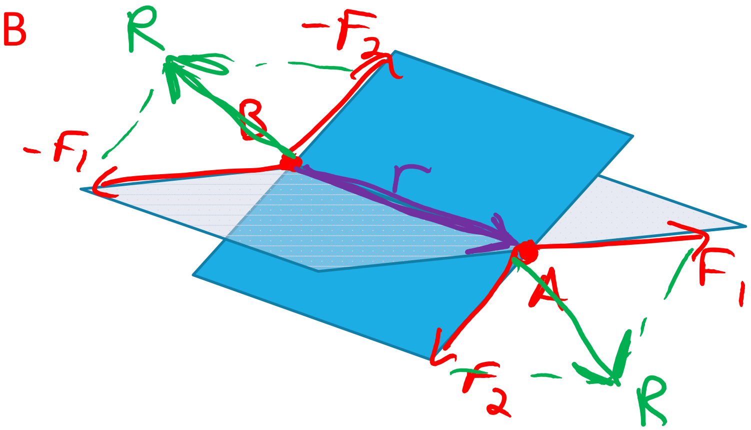
This means that when a couple acts on a rigid body, it does not matter where the two forces forming the couple act, or what direction and magnitude they have

The only thing which counts is the moment of the couple (magnitude & direction)

$$F_1 d_1 = F_2 d_2$$

3.14 Addition of couples

Consider two intersecting planes, P_1 & P_2. Assume that the 2 couples (**F_1** & **$-F_1$**, **F_2** & **$-F_2$**) are perpendicular to the line of intersection of the two planes and acting at A & B



It is clear that the resultant **R** of **F_1** & **F_2** and the resultant **–R** of **$-F_1$** & **$-F_2$** form a couple

3.14 Addition of couples

Denoting by **r** the vector joining B to A & recalling the definition of the moment of a couple, we express **M** of the resulting couple as follows:

$$\mathbf{M} = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2)$$

By Varignon's Theorem:

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$$

3.15 Couples can be represented by vectors

As was seen earlier, couples which have the same moment, whether they act in the same plane or parallel planes are equivalent

Therefore, it is not needed to draw the actual forces forming the couple

Instead, you can draw an arrow equal in magnitude and direction to the moment **M** of the couple

Also, if you have $\mathbf{M_1}$ & $\mathbf{M_2}$ of two couples, they can be represented by their resultant **M**, and the law of addition of vectors apply here

3.15 Couples can be represented by vectors

The vector representing a couple is called a couple vector

The symbol ↻ is added to avoid any confusion with vectors representing forces

A couple vector, like the moment of a couple, is a free vector

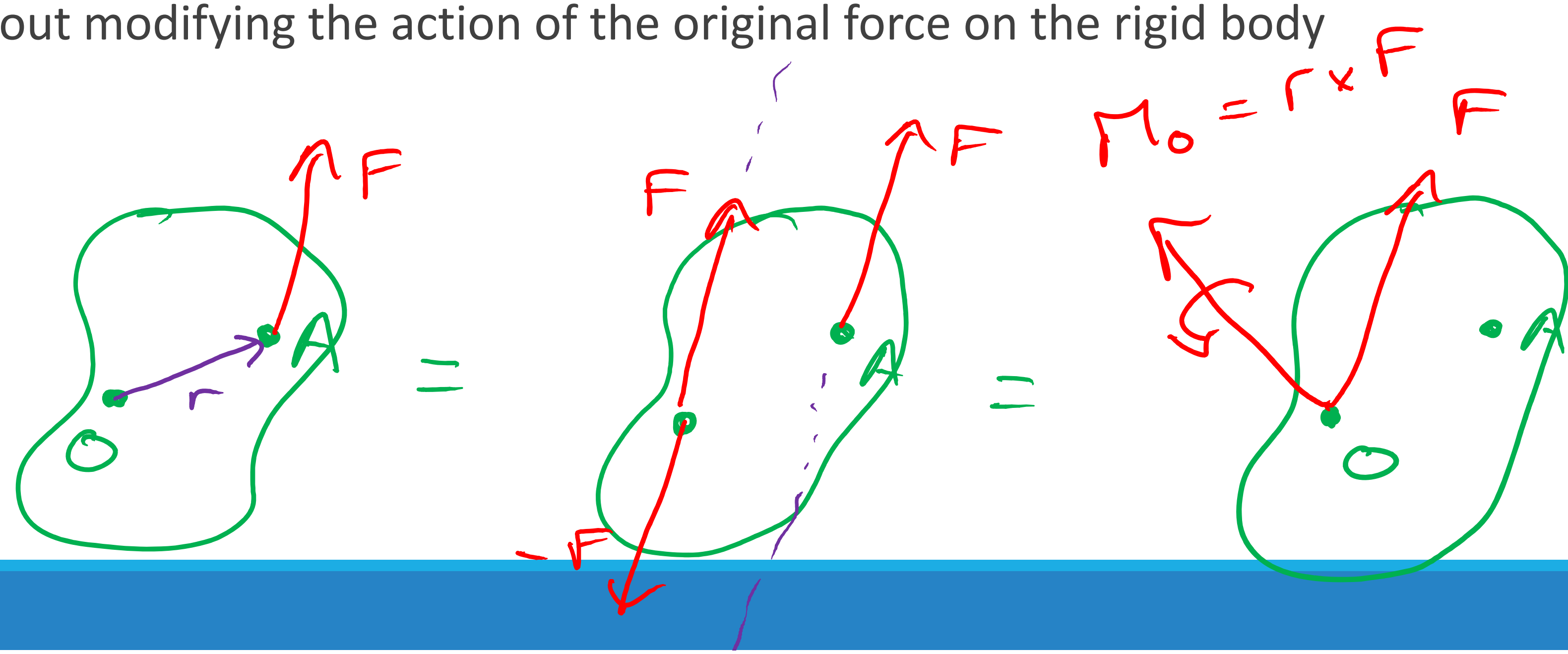
Furthermore, the couple vector **M** can be resolved into component vectors $\mathbf{M}_x$, $\mathbf{M}_y$ and $\mathbf{M}_z$ which are directed along the coordinate axes. These component vectors represent couples acting in the yz, zx, and xy planes, respectively.

3.16 Resolution of a given force into a force at O and a couple

$M = \hat{r} \times \hat{F} = Fd$

Consider a force **F** acting on a rigid body at point A defined by the position vector **r**

We can attach 2 forces at point O, one equal to **F** and the other equal to **–F**, without modifying the action of the original force on the rigid body



3.16 Resolution of a given force into a force at O and a couple

As a result of this transformation, a force **F** is now applied at O. The other two forces form a couple of moment $\mathbf{M_o} = \mathbf{rF}$.

Thus, any force **F** acting on a rigid body can be moved to an arbitrary point O provided that a couple is added whose moment is equal to the moment of **F** about O

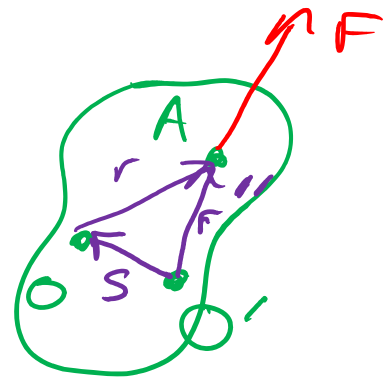
The couple is represented by a couple vector $\mathbf{M_o}$ perpendicular to the plane containing **r** and **F**. Since $\mathbf{M_o}$ is a free vector, it can be applied anywhere; for convenience, the couple is usually attached at O, together with **F**, and the combination obtained is referred to as a force-couple system.

3.16 Resolution of a given force into a force at O and a couple

If the force **F** had been moved to point O’, the moment $\mathbf{M}_{o'} = \mathbf{r'} \times \mathbf{F}$ of **F** about O’ should have been computed, and a new force-couple system consisting of **F** and $\mathbf{M}_{o'}$ would have been attached at O’

$$\mathbf{M}_{o'} = \mathbf{r'} \times \mathbf{F} = (\mathbf{r} + \mathbf{S}) \times \mathbf{F} = \mathbf{r} \times \mathbf{F} + \mathbf{S} \times \mathbf{F}$$

$$\mathbf{M}_{o'} = \mathbf{M}_{o} + \mathbf{S} \times \mathbf{F}$$



Where **S** is the vector joining O’ to O. **S** x **F** represents the moment about O’ of the force **F** applied at O.

3.16 Resolution of a given force into a force at O and a couple

The force-couple system obtained by transferring a force **F** from a point A to a point O consists of **F** and a couple vector $\mathbf{M_o}$ perpendicular to **F**

Conversely, any force-couple system consisting of a force **F** & a couple vector $\mathbf{M_o}$ which are mutually perpendicular can be replaced by a single equivalent force. This is done by moving the force **F** in the plane perpendicular to $\mathbf{M_o}$ until its moment about O is equal to the moment of the couple to be eliminated.

3.17 Reduction of a system of forces to one force and one couple

Consider a system of forces $\mathbf{F_1}$, $\mathbf{F_2}$, $\mathbf{F_3}$, …. acting on a rigid body at points A_1, A_2, A_3, …. defined by the position vectors $\mathbf{r_1}$, $\mathbf{r_2}$, $\mathbf{r_3}$, ….

As seen in the preceding section, $\mathbf{F_1}$ can be moved from A_1 to a given point O if a couple of moment $\mathbf{M_1}$ equal to the moment $\mathbf{r_1} \times \mathbf{F_1}$ of $\mathbf{F_1}$ about O is added to the original system of forces

Repeating this procedure with $\mathbf{F_2}$, $\mathbf{F_3}$, …. we obtain a system which consists of the original forces (now acting at O) and the added couple vectors

3.17 Reduction of a system of forces to one force and one couple

Since the forces are now concurrent, they can be added vectorially and replaced by their resultant **R**

Similarly, the couple vectors $\mathbf{M_1}$, $\mathbf{M_2}$, $\mathbf{M_3}$, …. can be added vectorially and replaced by a single couple vector $\mathbf{M_o^R}$

Therefore, any system of forces can be reduced to an equivalent force-couple system at a given point O

3.17 Reduction of a system of forces to one force and one couple

The equivalent force-couple system is defined by the equation:

$$\mathbf{R} = \Sigma \mathbf{F}; \qquad \mathbf{M}_o^R = \Sigma \mathbf{M}_o = \Sigma(\mathbf{r} \times \mathbf{F})$$

These equations show that the force **R** is obtained by adding all the forces of the system, while the moment of the resultant couple vector $\mathbf{M}_o^R$, called the moment resultant of the system, is obtained by adding the moments about O of all the forces of the system

3.17 Reduction of a system of forces to one force and one couple

Once a given system of forces has been reduced to a force & a couple at a point O, it can be easily reduced to a force and a couple at another point O'

Resultant **R** will remain unchanged, the new moment $\mathbf{M}_{o'}^{R}$ will be equal to the sum of $\mathbf{M}_{o}^{R}$ and the moment about O' of the force **R** attached at O

$$\mathbf{M}_{o'}^{R} = \mathbf{M}_{o}^{R} + \mathbf{S} \times \mathbf{R}$$

3.17 Reduction of a system of forces to one force and one couple

This reduction of a given force system to a single force **R** at O & a couple vector $\mathbf{M}_o^R$ can be carried out in terms of components

Resolving each position vector **r** & each force **F** of the system into rectangular components:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

3.17 Reduction of a system of forces to one force and one couple

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}$$

$$\mathbf{M_o}^\mathbf{R} = M_x^R\mathbf{i} + M_y^R\mathbf{j} + M_z^R\mathbf{k}$$

3.18 Equivalent system of forces

Two systems of forces are equivalent if they can be reduced to the same force-couple system at a given point O

In other words, two systems of forces, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, …. and $\mathbf{F'}_1$, $\mathbf{F'}_2$, $\mathbf{F'}_3$, …. , which act on the same rigid body are equivalent if, and only if, the sums of the forces and the sums of the moments about a given point O of the forces of the two systems are, respectively, equal.

$$\Sigma \mathbf{F} = \Sigma \mathbf{F'} \text{ and } \Sigma \mathbf{M}_o = \Sigma \mathbf{M'}_o$$

3.18 Equivalent system of forces

Resolving these forces & moments into their rectangular components, we can express the necessary & sufficient conditions for the equivalence of 2 systems of forces acting on a rigid body as follows:

$$\Sigma F_x = \Sigma F'_x \qquad \Sigma M_x = \Sigma M'_x$$

$$\Sigma F_y = \Sigma F'_y \qquad \Sigma M_y = \Sigma M'_y$$

$$\Sigma F_z = \Sigma F'_z \qquad \Sigma M_z = \Sigma M'_z$$

3.18 Equivalent system of forces

The physical significance these equations express is that the two systems of forces are equivalent if they tend to impart to the rigid body:

- the same translation in the x,y,z directions, and,
- the same rotation about the x,y,z axes

3.20 Further reduction of a system of forces

We saw that any given system of forces acting on a rigid body can be reduced to an equivalent force-couple system at O consisting of a force **R** and a couple vector $\mathbf{M}_o^R$

When **R** = 0, the force-couple system reduces to the couple vector $\mathbf{M}_o^R$. The given system of forces can then be reduced to a single couple, called the resultant couple of the system.

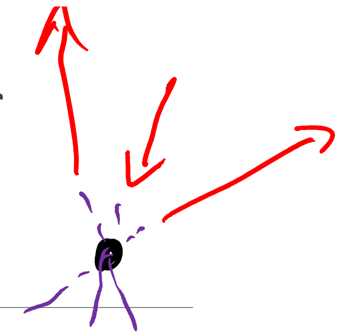
3.20 Further reduction of a system of forces

The systems of forces which can be reduced to a single force, or resultant, are the systems for which the force **R** and the couple vector $\mathbf{M}_o^R$ are mutually perpendicular

While this condition is generally not satisfied by systems of forces in space, it will be satisfied by systems consisting of:

- Concurrent forces,
- Coplanar forces, or
- Parallel forces

3.20 Further reduction of a system of forces

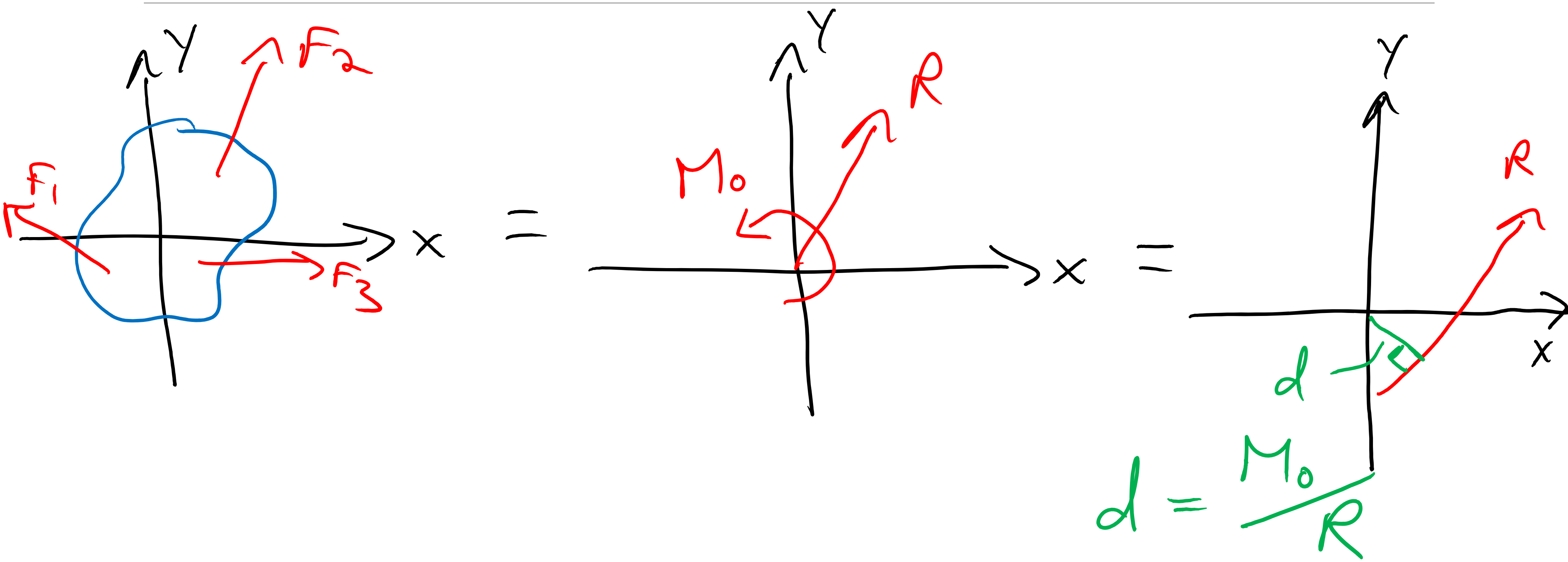


Concurrent forces are applied at the same point and can be added directly to obtain their resultant **R**. Thus, they always reduce to a single force.

Coplanar forces act in the same plane. The sum **R** of the forces will also lie in the same plane, but the moment of each force about O, and thus the moment resultant $\mathbf{M}_o^R$, will be perpendicular to that plane.

- Therefore, the force-couple system at O consists of a force **R** and a couple vector $\mathbf{M}_o^R$ which are mutually perpendicular. They can be reduced to a single force **R** by moving **R** until its moment about O becomes equal to $\mathbf{M}_o^R$.
- The distance from O to the line of action of **R** is $d = \mathbf{M}_o^R/\mathbf{R}$.

3.20 Further reduction of a system of forces



3.20 Further reduction of a system of forces

Therefore, the force-couple system at O is characterized by the components:

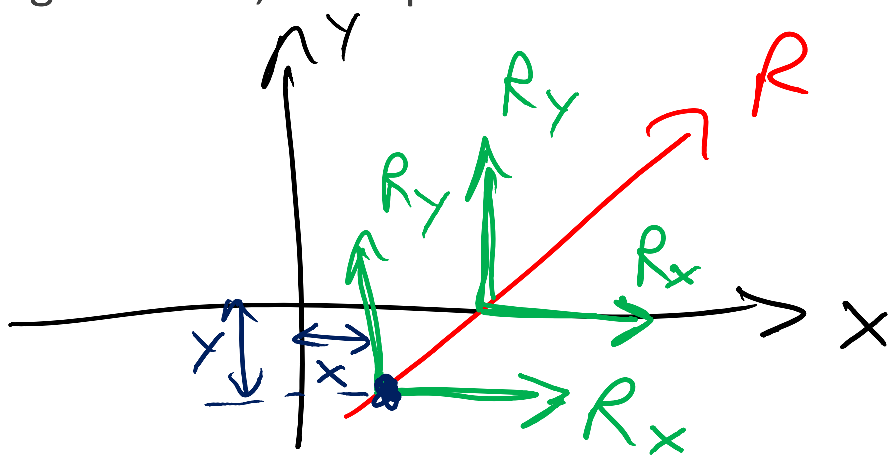
$R_x = \Sigma F_x$ $R_y = \Sigma F_y$ $M_z^R = M_o^R = \Sigma M_o$

To reduce the system to a single force **R**, we express that the moment of **R** about O must be equal to M_o^R:

$$xR_y - yR_x = M_o^R$$

If $y = 0$:

$$x = \frac{M_o^R}{R_y}$$

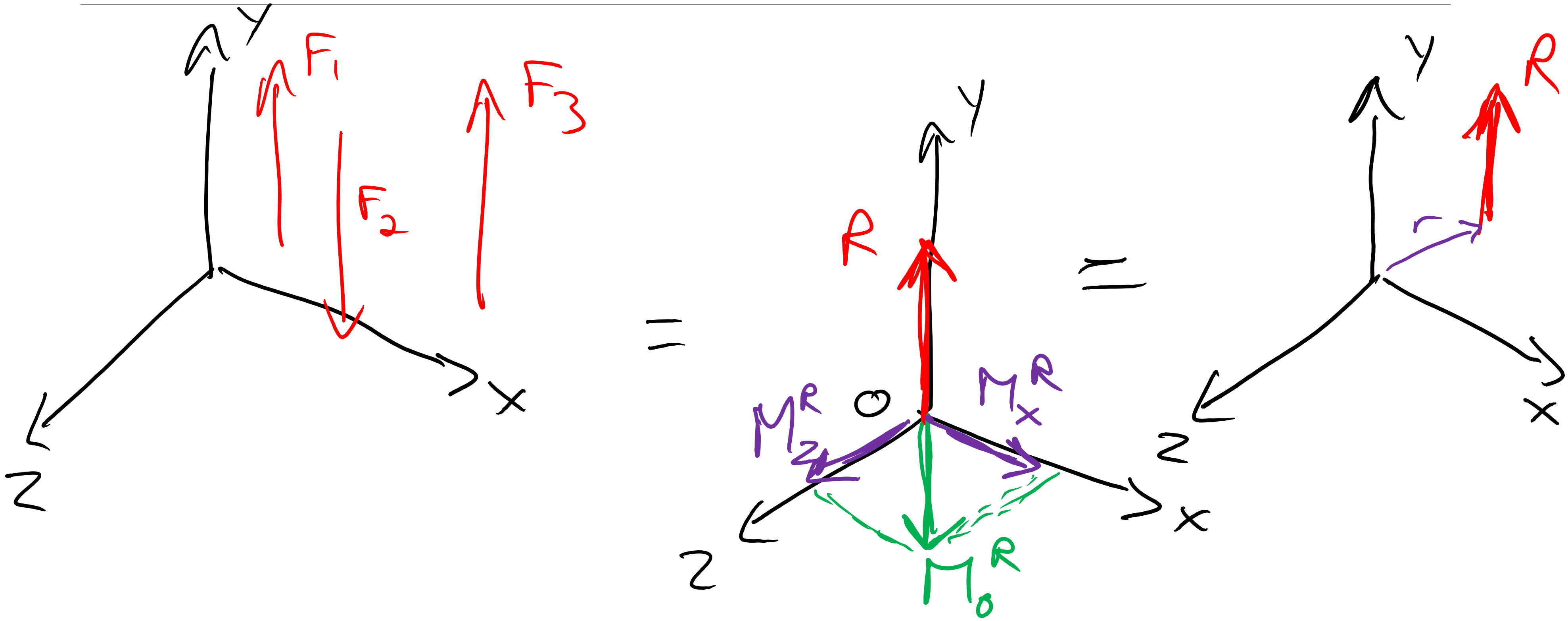


3.20 Further reduction of a system of forces

Parallel forces have parallel lines of action and may or may not have the same sense

- If these forces are parallel, say to the y-axis, then their resultant **R** should be parallel to the y-axis also
- Since the moment of a given force must be perpendicular to that force, then the moment of each force, or the moment resultant $\mathbf{M}_o^R$ will lie in the zx plane
- Therefore, the force-couple system at O consists of a force **R** and a couple vector $\mathbf{M}_o^R$ which are mutually perpendicular
- They can be reduced to a single force, or, if **R** = 0, to a single couple of moment $\mathbf{M}_o^R$

3.20 Further reduction of a system of forces



3.20 Further reduction of a system of forces

The force-couple system at O will be given by the components:

$R_y = \Sigma F_y$ $M_x^R = \Sigma M_x$ $M_z^R = \Sigma M_z$

The reduction of the system to a single force can be carried out by moving **R** to a new point of application A (x,0,z) chosen so that the moment of **R** about O is equal to $\mathbf{M_o^R}$:

$$\mathbf{r} \times \mathbf{R} = \mathbf{M_o^R}$$

$$(x\mathbf{i} + z\mathbf{k}) \times R_y\mathbf{j} = M_x^R\mathbf{i} + M_z^R\mathbf{k}$$

3.20 Further reduction of a system of forces

By computing the vector products & equating the coefficients of the corresponding unit vectors, we obtain 2 scalar equations which define the coordinates of A:

$-zR_y = M_x^R$ and $xR_y = M_z^R$

These equations express that the moments of **R** about the x and z axes must, respectively, be equal to M_x^R and M_z^R