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MAT 1330, Fall 2015 Assignment 2
Due Thursday October 1st by 8:00pm.
Late assignments will not be accepted; nor will unstapled assignments. Professors in the math department will not lend you a stapler; do not ask for one.

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Student Name ______________________ Student Number ______________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ______________________

QUESTION 1.

a) Solve the inequality $\frac{2}{x-3} > \frac{6}{2x+5}$.

The denominators are zero at $x=3$ and $x=-\frac{5}{2}$, respectively

Case 1: If $x>3$, then both denominators are positive.

Cross-multiply to get $2(2x+5)>6(x-3)$ or $x<14$

So we find $3<x<14$ is part of the solution

Case 2: If $-\frac{5}{2}<x<3$, then only $(x-3)$ is negative. Cross-multiply to get $2(2x+5)<6(x-3)$ or $x>14$

This case is incompatible. No solution here.

Case 3: If $x<-\frac{5}{2}$, both denominators are negative. Cross-multiply to get $2(2x+5)>6(x-3)$ or $x<14$ as in case 1.

Hence $x<-\frac{5}{2}$ is another part of the solution

Answer: $x<-\frac{5}{2}$ or $3<x<14$

b) Find the value(s) of $x \in [-\pi/2, \pi/2]$ for which $|\sin(2x)| = \cos(2x)$.

Since $\sin(2x)$ is an odd function, $|\sin(2x)|$ is even. Also, $\cos(2x)$ is even.
Therefore, it is enough to look at $x \in [0, \frac{\pi}{2}]$ and consider

$$\sin(2x) = \cos(2x)$$

But this is the case only if $2x = \frac{\pi}{4}$ so that $x = \frac{\pi}{8}$

By symmetry, $x = -\frac{\pi}{8}$ is another solution.

Answer: $x = \pm \pi/8$

c) Rationalize the numerator in the following expression $\dfrac{\sqrt{12} + \sqrt{x}}{\sqrt{12} - \sqrt{x}}$.

$$\frac{\sqrt{12} + \sqrt{x}}{\sqrt{12} - \sqrt{x}} = \frac{(\sqrt{12} + \sqrt{x})(\sqrt{12} - \sqrt{x})}{(\sqrt{12} - \sqrt{x})(\sqrt{12} - \sqrt{x})}$$

$$= \frac{12 - x}{12 - 2\sqrt{12x} + x}$$

Answer:

d) For which values of h does the equation $\dfrac{4x}{1+2x} = 3x+h$ have two distinct solutions for x?

Cross-multiplying, we get $4x = (3x+h)(1+2x)$

Expanding and sorting by powers of x, we get $6x^2 + (2h-1)x + h = 0$

The discriminant is $(2h-1)^2 - 24h = 4h^2 - 28h + 1$

The discriminant equals zero when $h = \frac{1}{8}\left[28 \pm \sqrt{28^2 - 16}\right]$

simplify: $h = \frac{1}{2}\left[7 \pm \sqrt{48}\right]$

Answer: $h < \frac{1}{2}\left[7 - \sqrt{48}\right]$ or $h > \frac{1}{2}\left[7 + \sqrt{48}\right]$

The discriminant is positive when $h > \frac{1}{8}\left[28 + \sqrt{28^2 - 16}\right]$

or when $h < \frac{1}{8}\left[28 - \sqrt{28^2 - 16}\right]$

QUESTION 2. Consider the discrete logistic equation $x_{t+1} = rx_t(1 - x_t/K)$.

(a) Calculate all steady states. [Hint: some of them may depend on parameters r, K.]

Solve: $x = rx\left(1 - \frac{x}{K}\right)$ one solution is $x = 0$

or $1 = r\left(1 - \frac{x}{K}\right) \Rightarrow x = K\frac{r-1}{r}$

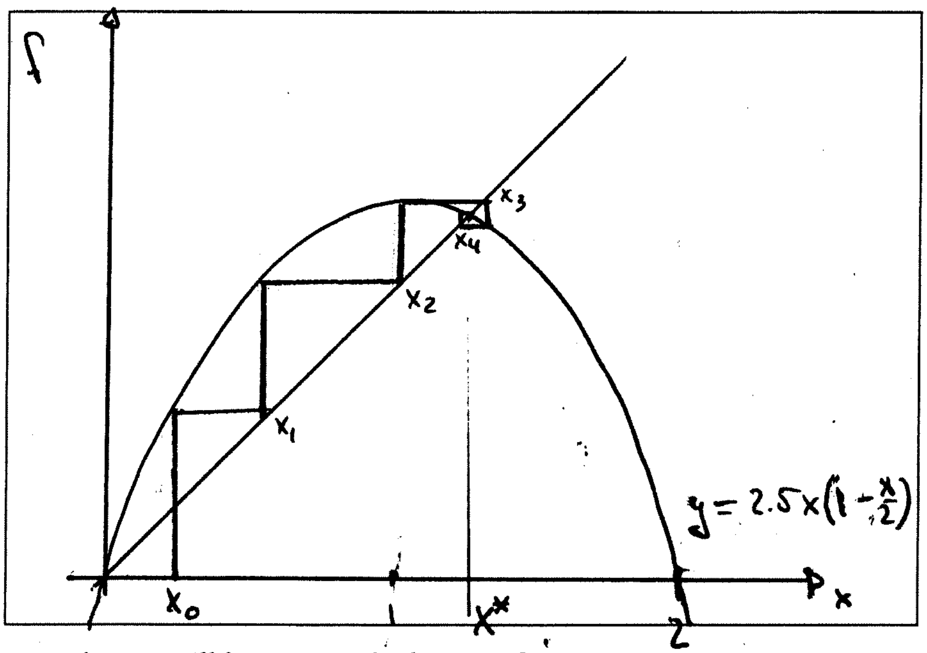
Answer: $x = 0$ or $x^* = K\frac{r-1}{r}$

(b) From now on, set $r = 2.5$ and $K = 2$. Start with $x_0 = 0.2$ and calculate x_1, x_2, x_3, x_4.

Answer: $x_1 = 0.45$, $x_2 = 0.87$, $x_3 = 1.23$, $x_4 = 1.18$

(c) Draw the graph of the updating function and start the cobwbbing process at $x_0 = 0.2$.

Answer:



(d) What do you observe will happen in the long run?

Looks like the solution will approach x^*.

QUESTION 3. Island Biogeography asks the question of how many species are present on an island. Obviously, the answer depends on how many individuals arrive on the island and how many species become extinct. In 1970, Simberloff and Wilson conducted a large experiment in which they counted insect species in mangrove islands after complete defaunation. We explain a corresponding model.

Denote by X_t the number of species (not individuals) on the island at the beginning of month t. If we assume that each species has the same, constant probability of becoming extinct in a given month, then the number of species that become extinct from one month to the next is a linearly increasing function of the number of species present, say $E_t = 0.2X_t$. The number of new species immigrating decreases with X_t since the more species there are on the island, the smaller the chance that any newly arriving individual is actually a new species. Therefore, we set immigration to be $I_t = 10 - 0.2X_t$. The number of species on the island next month is then the number of species this month plus immigrations minus extinctions.

(a) The DTDS for the number of species is $X_{t+1} = \boxed{X_t + 10 - 0.2X_t - 0.2X_t = 10} + 0.6X_t$

(b) Calculate X_1, X_2, X_3 if $X_0 = 0$.

Answer: $\boxed{X_1 = 10,\quad X_2 = 16,\quad X_3 = 19.6,}$

(c) Calculate the steady state $X^* = \boxed{\frac{10}{1-0.6} = \frac{10}{0.4} = 25}$

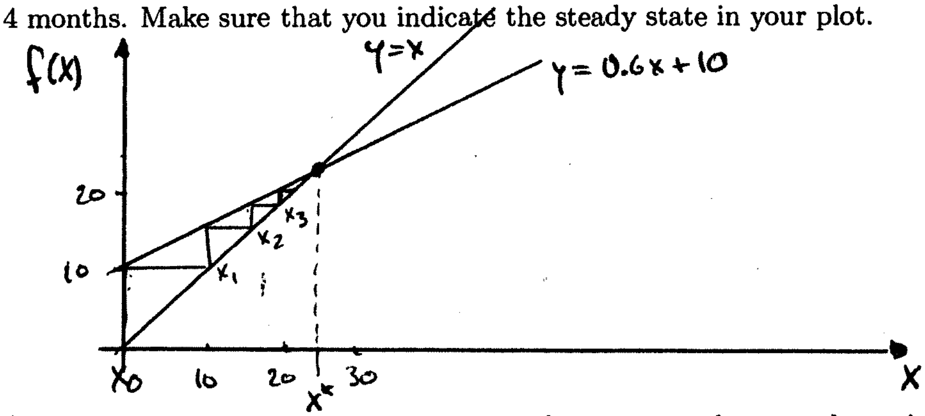
(d) Is the steady state stable of unstable?

Answer: $\boxed{\text{Stable, since } 0<0.6<1}$ and the theorem in class.

(e) How long will it take for the number of species to grow from complete defaunation to 90% of the steady-state value?

Answer: $\boxed{t = 4.5. \text{ So after 5 month}}$ we have over 90% of the steady state.

(f) Draw a graph of the updating function and start the cobwebbing process from $X_0 = 0$ for 4 months. Make sure that you indicate the steady state in your plot.



(g) Through conservation measures, researchers want to decrease the extinction rate of species on the island so that $E_t = eX_t$ for some $0 < e < 0.2$. How small must e be so that the steady state would be $X^* = 30$?

Answer: $\boxed{e \text{ has to be } \frac{2}{15} \approx 0.133}$ (or smaller)

[Use the back of the page for calculations.]

Details for 3

(e) The general solution: $X_t = r^t(X_0 - X^*) + X^* \overset{\text{if } X_0 = 0}{=} X^*(1 - r^t)$

The condition: $X_t = 0.9 \cdot X^*$.

The equation for t: $0.9X^* = X^*(1 - r^t) = X^*(1 - (0.6)^t)$

Cancel X^* and isolate t!

$$(0.6)^t = 0.1$$

$$t = \frac{\ln 0.1}{\ln 0.6} \approx 4.5$$

(d) If $E_t = eX_t$ then

$$X_{t+1} = 10 + (1 - 0.2 - e)\,X_t$$

This DTDS has steady state $X^* = \frac{10}{1 - (1 - 0.2 - e)} = \frac{10}{0.2 + e}$

The condition $X^* = 30 = \frac{10}{0.2 + e}$ gives

$$e = \frac{10}{30} - 0.2 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \approx 0.1333$$