Solutions to Assignment 1: MAT1330

Question 1:

(a) $\frac{5}{x-2} > 3$

Case 1: $x - 2 > 0$, which is the case when $x > 2$. We get:

$$5 > 3(x-2)$$

which leads to

$$\frac{11}{3} > x.$$

So $x \in (2, \frac{11}{3})$ is the solution to this case.

Case 2: $x - 2 < 0$, which is the case when $x < 2$. Then $5 < 3(x-2)$ leads to the contradiction $\frac{11}{3} < x$ but we had assumed $x < 2$. So we have no solution in this case.

The solution hence comes from case 1 and is given by $x \in (2, \frac{11}{3})$.

(b) $|x^2 + 2x - 12| = 1$. Evaluating the absolute value function hence gives $x^2 + 2x - 12 = \pm 1$ which leads to two cases:

Case 1: $x^2 + 2x - 12 = 1$, which gives solutions $x_{1,2} = -1 \pm \sqrt{14}$.

Case 2: $x^2 + 2x - 12 = -1$ which gives solutions $x_{3,4} = -1 \pm \sqrt{12}$.

Solution: $x \in \{-1 - \sqrt{14}, -1 - \sqrt{12}, -1 + \sqrt{12}, -1 + \sqrt{14}\}$.

Question 2:

(a) We have a DTDS.

x_t : population density

time step: one year

updating function: $x_{t+1} = 3x_t \cdot e^{-x_t/2}$.

If we only measure every other year, this is a two-time step. So we have to compute the composition of the updating function with itself:

$$(f \circ f)(x) = f(f(x)) = 3 \cdot (3x \cdot e^{-x/2}) \cdot e^{-x/2} = 9 \cdot x \cdot e^{-\frac{x}{2}} \cdot e^{-\frac{3}{2}xe^{-x/2}}.$$

(b) $x_{t+1} = g(x_t) = \frac{5x_t}{2+\frac{x_t}{3}}$.

The previous time step is asked, so we have to compute the inverse of the updating function and then evaluate at step $t = 10$.

Updating function: $y = f(x) = \frac{5x}{2+\frac{x}{3}}$. The inverse can be computed via

$$y = \frac{5x}{2 + \frac{x}{3}}$$

$$\vdots$$
$$x = \frac{6y}{15-y}.$$

So $f^{-1}(y) = \frac{6y}{15-y}$. It is left to evaluate at step $t = 10$:
$x_9 = \frac{6x_{10}}{15-x_{10}}$.

Question 3:

Another DTDS. Apply 10g of sunscreen every hour. So $c = 10$. The evaporation rate (or wash away) is 30 percent, so we get $r = 0.7$ as the residue of sunscreen on the skin.

(a) $x_{t+1} = 0.7 \cdot x_t + 10$, $f(x) = 0.7 \cdot x + 10$.

(b) We have $r \neq 1$, so we have as the general solution formula:

$$x_t = r^t(x_0 - x^*) + x^*$$

where $x^* = \frac{c}{1-r} = \frac{10}{0.3} = \frac{100}{3}$. Hence

$$x_t = -\frac{100}{3} \cdot 0.7^t + \frac{100}{3}.$$

(c) The question reads: ”you may assume you had no sunscreen at all when you woke up”. Hence $x_0 = 0$. $x_8 = -\frac{100}{3} \cdot 0.7^8 + \frac{100}{3} = 31.41$.

Question 4:

Increase by 20 percent every day. So $r = 1.2$ and the time step is **one day**. The initial volume is 1 cm^3, so $x_0 = 1$.

(a) $x_{t+1} = 1.2x_t$.

(b) Here we have $c = 0, r = 1.2$. So $r \neq 1$ and we get as general solution formula:

$$x_t = 1.2^t(x_0 - x^*) + x^*$$

where $x^* = \frac{0}{1-1.2} = 0$. Hence

$$x_t = 1.2^t.$$

It is asked when the volume is 1.8 cm^3 which translates to 'for what t is $x_t = 1.8$?'

$$1.8 = 1.2^t$$
$$\vdots$$
$$t = \frac{\ln(1.8)}{\ln(1.2)} = 3.22.$$

Now we have discrete time steps, so it takes 4 days until we measure a volume of size at least 1.8 cm^3.

(c) When does it double in volume? Write time to double as k. We have $x_0 = 1$.

$$x_{t+k} = 2x_t = 2(1.2^t).$$

On the other hand, $x_{t+k} = 1.2^{t+k}$. So

$$1.2^{t+k} = 2 \cdot (1.2^t)$$

We see here from cancelling that we could have also assumed $x_k = 2 \cdot 1.2^t$.

$$1.2^k = \ln 2$$

giving us

$$k = \frac{\ln 2}{\ln(1.2)} = 3.8.$$

Answer: it takes 4 days for the volume to double.

(d) The container is half full on day 10. From (c) we know it takes 4 days to double in volume, so it will be full on day $10 + 4 = 14$.

To check the volume of the container, we hence need x_{14}.

$$x_{14} = 1.2^{14} = 12.84.$$

Answer: The container has a volume of 12.84 cm^3.