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CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number		Section(s)
Mathematics	206/2		All
Examination	**Date**	**Time**	**Pages**
Final	December 2013	3 Hours	2

Instructors	**Course Examiner**
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Special Instructions

▷ **Only approved calculators are allowed.**

MARKS

[4] 1. Simplify the expressions below. Do not use a calculator.

(a) $-\sqrt{18}+2\sqrt{8}+\sqrt{72}$ (b) $\frac{2}{3}\ln 8 - \ln(5^2-1)$

[4] 2. Rationalize the denominator:

(a) $\dfrac{\sqrt{2}}{\sqrt{3}-3\sqrt{2}}$ (b) $\dfrac{2-\sqrt{5}}{2+3\sqrt{5}}$

[6] 3. Simplify the expressions:

(a) $4x^2(4x^3-3x^2-2x)-5x(4x^4-3x^3+2x^2+4x)$ (b) $\dfrac{x^2+4x+4}{x^2+3x+2}$

[8] 4. Factor the polynomials completely:

(a) $6x^2+5x+1$ (b) $8x^3+27$

[4] 5. Perform the arithmetic operations and simplify:

$$\frac{x}{x-3}-\frac{x+1}{x^2+5x-24}$$

[9] 6. Solve the equations:

(a) $\dfrac{x}{x^2-9}+\dfrac{4}{x+3}=\dfrac{3}{x^2-9}$ (b) $\log_5(x+3)=1-\log_5(x-1)$

(c) $8^{-x+14}=(16)^x$

[8] 7. Solve the inequalities, express your answer using set notation or interval notation:

(a) $0<\dfrac{3x+2}{2}<4$ (b) $\left|\dfrac{2x+3}{3}-\dfrac{1}{2}\right|<1$

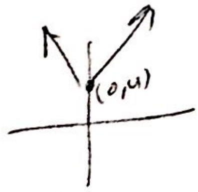
[4] **8.** Solve the system of equations:

Note: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$\begin{aligned} x^2 + y^2 &= 36 \\ x + y &= 8 \end{aligned}$$

[8] **9.** (a) Which of the points $A(2,3)$, $B(6,4)$ is closer to the point $C(4,3)$?

(b) Show that the equation $2x^2 + 2y^2 + 8x + 7 = 0$ represents a circle. Find coordinates of the center and radius of the circle.



[6] **10.** Find the domain and range of the functions (do not graph):

(a) $f(x) = \dfrac{3x}{x^2 - 4}$ (b) $g(x) = \sqrt{-x-2}$ (c) $h(x) = |x| + 4$

$\dfrac{2}{|x|}$

[5] **11.** Sketch the graph of the function $f(x) = \log(x-4) + 2$, starting from the graph of the function $g(x) = \log x$ and using appropriate transformations.

✓ [8] **12.** Let $f(x) = \dfrac{x}{x-1}$ and $g(x) = \dfrac{-4}{x}$. Find:

(a) fg (b) $\dfrac{f}{g}$ (c) $f \circ g$ (d) $g \circ f$

[8] **13.** (a) Find the inverse of the function $f(x) = \dfrac{2x}{3x-1}$.

~~(b)~~ Find the vertical and horizontal asymptotes of both f and f^{-1} above.

✓ [5] **14.** The diagonal of a rectangle measures 10 inches. If the length is 2 inches more than the width, find the dimensions of the rectangle.

[5] **15.** From each corner of a square piece of sheet metal, remove a square of side 9 centimeters. Turn up the edges to form an open box. If the box is to hold 144 cubic centimeters, what should be the dimensions of the piece of sheet metal?

[8] **16.** The number N of bacteria present in a culture at time t (in hours) obeys the law of uninhibited growth

$$N(t) = 1000e^{0.01t}$$

(a) Determine the number of bacteria at $t = 0$ hours.

(b) What is the population after 4 hours?

(c) When will the number of bacteria double?

December 2013 Final Exam Solutions

① a) $-\sqrt{18} + 2\sqrt{8} + \sqrt{72}$

$-\sqrt{9\cdot 2} + 2\sqrt{4\cdot 2} + \sqrt{8\cdot 9}$

$-3\sqrt{2} + 4\sqrt{2} + 3\sqrt{2\cdot 4}$

$\sqrt{2} + 6\sqrt{2}$

$7\sqrt{2}$

b) $\frac{2}{3}\ln 8 - \ln(5^2 - 1)$

$\frac{2}{3}\ln 8 - \ln(25 - 1)$

$\frac{2}{3}\ln 8 - \ln 24$

$\ln(8)^{2/3} - \ln 24$

$\ln 4 - \ln 24$

$\ln\left(\frac{4}{24}\right)$

$\ln\left(\frac{1}{6}\right)$

BEDMAS
- deal with brackets first!

Brackets
Exponents
Division
Multiplication
Addition
Subtraction

② a) $\frac{\sqrt{2}}{\sqrt{3} - 3\sqrt{2}} \cdot \frac{\sqrt{3} + 3\sqrt{2}}{\sqrt{3} + 3\sqrt{2}}$

$\frac{\sqrt{2}(\sqrt{3} + 3\sqrt{2})}{3 - 9(2)}$

$\frac{\sqrt{6} + 3(2)}{-15}$

$\frac{6 + \sqrt{6}}{-15}$

$-\frac{2}{5} - \frac{\sqrt{6}}{15}$

2)

b) $\dfrac{2-\sqrt{5}}{2+3\sqrt{5}} \cdot \dfrac{2-3\sqrt{5}}{2-3\sqrt{5}}$

$$\frac{(2-\sqrt{5})(2-3\sqrt{5})}{4-9(5)}$$

$$\frac{4-6\sqrt{5}-2\sqrt{5}+3\sqrt{25}}{-41}$$

$$\frac{4-8\sqrt{5}+3(5)}{-41}$$

$$\frac{19-8\sqrt{5}}{-41}$$

3 a) $4x^2(4x^3-3x^2-2x)-5x(4x^4-3x^3+2x^2+4x)$

$16x^5-12x^4-8x^3-20x^5+15x^4-10x^3-20x^2$

$-4x^5+3x^4-18x^3-20x^2$

b) $\dfrac{x^2+4x+4}{x^2+3x+2}$

product	1, 4	2, 2
sum	5	4

product	1, 2
sum	3

factor numerator: x^2+4x+4

$x^2+2x+2x+4$

$x(x+2)+2(x+2)$

$(x+2)(x+2)$

thus,

factor denominator: x^2+3x+2

$x^2+2x+x+2$

$x(x+2)+(x+2)$

$(x+2)(x+1)$

$$\frac{\cancel{(x+2)}(x+2)}{\cancel{(x+2)}(x+1)} = \boxed{\frac{x+2}{x+1} \quad \text{where } x \neq -2,\ x \neq -1}$$

④ a) $6x^2 + 5x + 1$

<u>check:</u> is it prime? $b^2 - 4ac = 5^2 - 4(6)(1) = 1$

Since $b^2 - 4ac =$ positive value (not 0) there is real solutions

<u>can</u> be factored

product	6, 1	3, 2
sum	7	5

$6x^2 + 5x + 1$

$6x^2 + 3x + 2x + 1$

$3x(2x+1) + (2x+1)$

$\boxed{(2x+1)(3x+1)}$

b) $8x^3 + 27$

$2^3x^3 + 3^3$

$(2x)^3 + 3^3$ → this is a sum of cubes

$\boxed{(2x+3)\underbrace{(4x^2 - 6x + 9)}}$

can this be factored further?

$2x + 3$

$4x^2 \quad 6x \quad 9$

$(2x+3)(4x^2 - 6x + 9)$

$= 8x^3 - 12x^2 + 18x + 12x^2 - 18x + 27$

$= 8x^3 + 27$

No, because:

$b^2 - 4ac \longrightarrow (-6)^2 - 4(4)(9) = -108$

since $b^2 - 4ac < 0$ this trinomial is prime (can't be factored further)

⑤ $\dfrac{x}{x-3} - \dfrac{x+1}{x^2 + 5x - 24}$

product	6, −4	−2, 12	2, −12	3, +8
sum	2	10	−10	5

$x^2 + 5x - 24$

$x^2 + 8x - 3x - 24$

$x(x+8) - 3(x+8)$

$(x+8)(x-3)$

$\dfrac{x}{(x-3)} - \dfrac{x+1}{(x-3)(x+8)}$

$\dfrac{x(x+8) - (x+1)}{(x-3)(x+8)}$

$\dfrac{x^2 + 8x - x - 1}{(x-3)(x+8)} = \dfrac{x^2 + 7x - 1}{(x-3)(x+8)}$

K Kefalas

6

a) $\dfrac{x}{x^2-9} + \dfrac{4}{x+3} = \dfrac{3}{x^2-9}$ where $x \neq -3$, $x \neq 3$

$\dfrac{x}{(x-3)(x+3)} + \dfrac{4}{(x+3)} = \dfrac{3}{(x-3)(x+3)}$ } multiply the entire expression by $(x-3)(x+3)$

$$x + 4(x-3) = 3$$
$$x + 4x - 12 = 3$$
$$5x = 15$$
$$x = 3$$

But $x \neq -3$, $x \neq 3$; so NO solution.

b) $\log_5(x+3) = 1 - \log_5(x-1)$

The domain is $x+3>0$, $x>-3$; $x-1>0$, $x>1$

so our solution must satisfy $x>1$

$$\log_5(x+3) = 1 - \log_5(x-1)$$
$$\log_5(x+3) + \log_5(x-1) = 1$$
$$\log_5(x+3)(x-1) = 1$$
$$5^1 = (x+3)(x-1)$$
$$5 = x^2 - x + 3x - 3$$
$$0 = x^2 + 2x - 8$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \rightarrow \frac{-2 \pm \sqrt{4-4(1)(-8)}}{2} = \frac{-2 \pm \sqrt{36}}{2} = \frac{-2 \pm 6}{2}$$

$$x = -4, 2$$

Since $x>1$, our solution is $x=2$.

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c) $8^{-x+14} = 16^{x}$

$(2^3)^{-x+14} = (2^4)^{x}$

$(2)^{-3x+42} = (2)^{4x}$

hence, $-3x+42 = 4x$

$42 = 7x$

$\boxed{x=6}$

Check: $8^{-6+14} = 16^{6}$

$8^{8} = 16^{6}$

$16777216 = 16777216$ OK

7 a) $0 < \frac{3x+2}{2} < 4$

$0 < \frac{3x+2}{2}$

$0 < 3x+2$

$-2 < 3x$

$-\frac{2}{3} < x$

$\frac{3x+2}{2} < 4$

$3x+2 < 8$

$3x < 6$

$x < 2$

thus $\boxed{-\frac{2}{3} < x < 2}$

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b) $\left|\frac{2x+3}{3} - \frac{1}{2}\right| < 1$

$\boxed{\text{thus, } -1 < x < 3/4}$

1 $+\left(\frac{2x+3}{3} - \frac{1}{2}\right) < 1$

$2(2x+3) - 3 < 6$

$4x+6-3 < 6$

$4x < 3$ so $\boxed{x < 3/4}$

2 $-\left(\frac{2x+3}{3} - \frac{1}{2}\right) < 1$

$\left(\frac{2x+3}{3} - \frac{1}{2}\right) > -1$

$2(2x+3) - 3 > -1$

$4x+6-3 > -1 \longrightarrow 4x+3 > -1 \rightarrow \boxed{x > -1}$

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K Kefalas .6

8) Solve

$$x^2 + y^2 = 36$$
$$x + y = 8 \longrightarrow y = 8 - x$$

$$x^2 + (8-x)^2 = 36$$
$$x^2 + (8-x)(8-x) = 36$$
$$x^2 + 64 - 16x + x^2 = 36$$
$$2x^2 - 16x + 28 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{16 \pm \sqrt{(-16)^2 - 4(2)(28)}}{4}$$

$$x = \frac{16 \pm \sqrt{32}}{4} = 4 \pm \frac{\sqrt{8 \cdot 4}}{4}$$

$$x = 4 \pm \frac{2\sqrt{8}}{4}$$

$$\sqrt{8} = \sqrt{2 \cdot 4} = 2\sqrt{2}$$

$$x = 4 \pm \frac{4\sqrt{2}}{4} = 4 \pm \sqrt{2}$$

solutions

If $x = 4 + \sqrt{2}$ then $y = 8 - x = 8 - 4 - \sqrt{2} = 4 - \sqrt{2}$

If $x = 4 - \sqrt{2}$ then $y = 8 - x = 8 - 4 + \sqrt{2} = 4 + \sqrt{2}$

<u>Check</u> $x^2 + y^2 = 36$
$x + y = 8$

• for $x = 4 + \sqrt{2}$; $y = 4 - \sqrt{2}$

$$x^2 + y^2 = 36$$
$$(4+\sqrt{2})^2 + (4-\sqrt{2})^2 = 36 \quad \underline{OK}$$

$$x + y = 8$$
$$4 + \sqrt{2} + 4 - \sqrt{2} = 8 \quad \underline{OK}$$

• for $x = 4 - \sqrt{2}$; $y = 4 + \sqrt{2}$

$$x^2 + y^2 = 36$$
$$(4-\sqrt{2})^2 + (4+\sqrt{2})^2 = 36$$
<u>OK</u>

$$x + y = 8$$
$$4 - \sqrt{2} + 4 + \sqrt{2} = 8 \quad \underline{OK}$$

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9)

a) distance $= \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$

for point A (2,3)
point C (4,3)

distance $= \sqrt{(2-4)^2 + (3-3)^2} = \underline{\underline{2}}$

for point B (6,4)
point C (4,3)

distance $= \sqrt{(6-4)^2 + (4-3)^2} = \sqrt{5} \approx 2.2$

therefore, point A is closer to point C.

b) $2x^2 + 2y^2 + 8x + 7 = 0$

$\underbrace{2x^2 + 8x}_{\text{complete the square}} + 2y^2 = -7$

$2(x^2 + 4x) + 2y^2 = -7$

$2(x^2 + 4x + 4) + 2y^2 = -7 + 8$

$2(x+2)^2 + 2y^2 = 1$

$2(x+2)^2 + 2(y-0)^2 = 1$ ← Circle must take the form:
$(x-h)^2 + (y-k)^2 = r^2$

$(x+2)^2 + (y-0)^2 = 1/2$ $\quad (x-(-2))^2 + (y-0)^2 = 1/2$

⟶ center $(h,k) = (-2,0)$

⟶ Since $r^2 = 1/2$, then $r = \sqrt{1/2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

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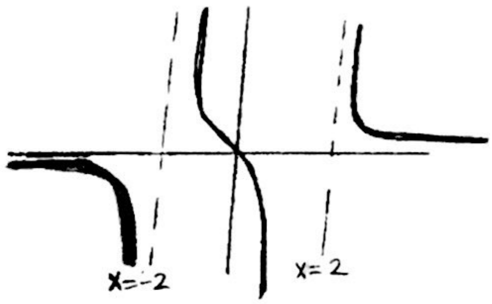
10) a) (you can graph to solve it, but no marks given for graph)

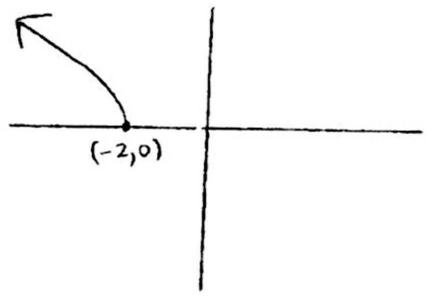
$f(x) = \frac{3x}{x^2-4} = \frac{3x}{(x-2)(x+2)}$



domain: $\{x \mid x \neq -2, x \neq 2\}$
Range: $y \in \mathbb{R}$ (all real numbers)
$y \in (-\infty, \infty)$

* to find the range, best to graph it
• this problem you need to graph $f(x) = \frac{3x}{x^2-4}$ } 8 step procedure or do a rough sketch

b) $g(x) = \sqrt{-x-2}$
$g(x) = \sqrt{-(x+2)}$

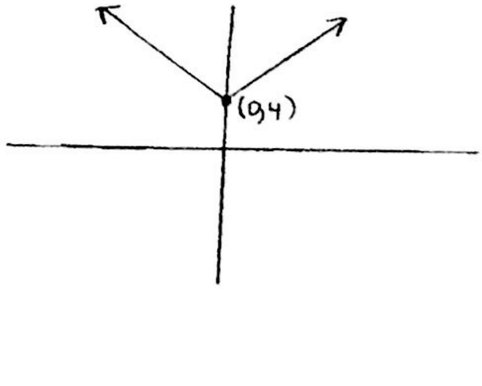


for domain, $-x-2 \geq 0$
$-x \geq 2$
$x \leq -2$

so $x \in R$ from $(-\infty, -2]$

Range: $y \in R$ from $[0, \infty)$

c) $h(x) = |x| + 4$



domain: $x \in R$ $(-\infty, \infty)$
Range: $y \in R$ $[4, \infty)$

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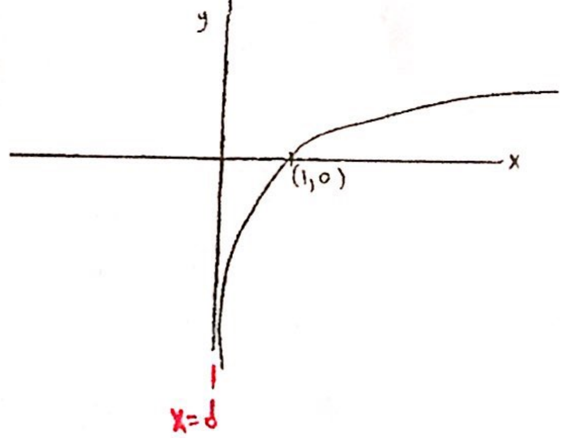
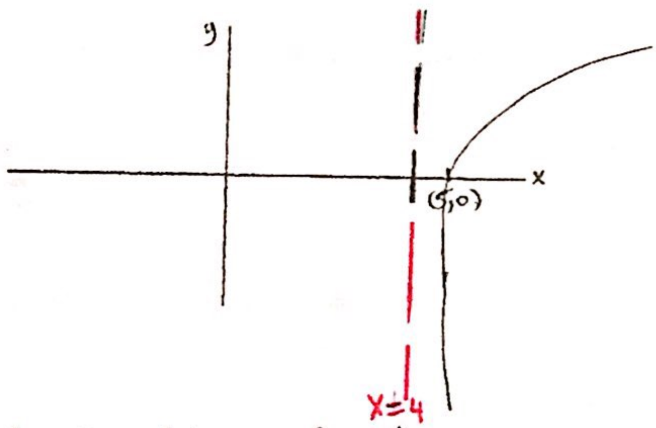
11) start with $g(x) = \log x$



asymptote at $x=0$

domain: $x > 0$

step 2: $g(x) = \log(x-4)$



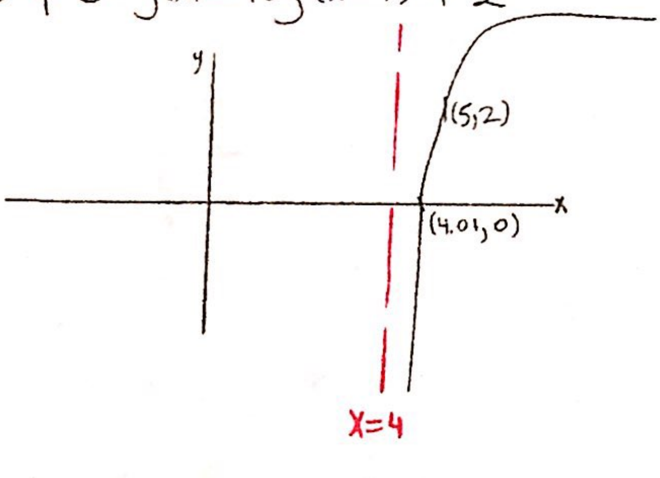
shifts 4 units to the right

$x=4$ is a vertical asymptote

$\log(4-4) = \log(0)$

↳ undefined

step 3: $g(x) = \log(x-4) + 2$



shifts up 2 units vertically

$(5,0) \rightarrow$ becomes $(5,2)$

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⑫ let $f(x) = \dfrac{x}{x-1}$

$g(x) = -\dfrac{4}{x}$

find
a) $fg \rightarrow f(x)\,g(x)$
b) $f/g \rightarrow f(x)\,/\,g(x)$
c) $f \circ g \rightarrow f(g(x))$
d) $g \circ f \rightarrow g(f(x))$

a) $f(x)g(x) = \left(\dfrac{x}{x-1}\right)\left(\dfrac{-4}{x}\right) = \dfrac{-4}{x-1}$

domain: $\{x \mid x \neq 0,\ x \neq 1\}$

b) $\dfrac{f}{g} = \dfrac{f(x)}{g(x)} = \dfrac{x}{x-1} \div \dfrac{-4}{x} = \dfrac{x^2}{-4(x-1)}$

domain: $\{x \mid x \neq 0,\ x \neq 1\}$

c) $f \circ g = f(g(x))$
$f\left(\dfrac{-4}{x}\right)$

$$\frac{-\frac{4}{x}}{\frac{-4}{x} - 1} = \frac{-4}{x} \div \frac{-4-x}{-1}$$

$$= -\frac{4}{x} \cdot \frac{-1}{-4-x} = \frac{4}{(x)(-4-x)}$$

$x \neq 0$
$-4-x \neq 0$
$-4 \neq x$

fog's
domain: $\{x \mid x \neq -4,\ x \neq 0\}$

d) $g(f(x)) \rightarrow g\left(\dfrac{x}{x-1}\right)$

$$\frac{-4}{\left(\frac{x}{x-1}\right)} = \frac{-4(x-1)}{x} = \frac{-4x+4}{x} = -4 + \frac{4}{x}$$

domain of $g \circ f$ $(g(f(x)))$: $\{x \mid x \neq 0,\ x \neq 1\}$

13 $\boxed{f(x) = \dfrac{2x}{3x-1}}$

a) $x = \dfrac{2y}{3y-1}$

$x(3y-1) = 2y$

$3xy - x = 2y$

$3xy - 2y = x$

$y(3x-2) = x$

$\boxed{f^{-1}(x) = y = \dfrac{x}{3x-2}}$

verify that ① $f(f^{-1}(x)) = x$ and ② $f^{-1}(f(x)) = x$:

① $f(f^{-1}(x)) = f\left(\dfrac{x}{3x-2}\right) = \dfrac{2\left(\frac{x}{3x-2}\right)}{3\left(\frac{x}{3x-2}\right)-1} = \dfrac{\frac{2x}{3x-2}}{\frac{3x-(3x-2)}{3x-2}}$

$= \dfrac{2x}{3x-2} \div \dfrac{2}{3x-2}$

$= \dfrac{2x}{2}$

$= x$

domain of $f \circ f^{-1}$: $\{x \mid x \neq 2/3\}$

② $f^{-1}(f(x)) = f^{-1}\left(\dfrac{2x}{3x-1}\right) = \dfrac{\frac{2x}{3x-1}}{3\left(\frac{2x}{3x-1}\right)-2} = \dfrac{\frac{2x}{3x-1}}{\frac{6x-2(3x-1)}{3x-1}} = \dfrac{2x/(3x-1)}{\frac{6x-6x+2}{3x-1}}$

$= \dfrac{\frac{2x}{3x-1}}{\frac{2}{3x-1}}$

domain of $f^{-1} \circ f$: $\{x \mid x \neq 1/3\}$

$= \dfrac{2x}{2}$

thus, verification is succesful meaning $f^{-1}(x)$ solved $= x$

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b) $f(x) = \dfrac{2x}{3x-1}$ Vertical asymptote $x = 1/3$

$f^{-1}(x) = \dfrac{x}{3x-2}$ vertical asymptote $x = 2/3$

since improper, use long division:

$$3x-1 \overline{\smash{)}\,2x} \quad \frac{2}{3}$$
$$-(2x - 2/3)$$
$$2/3$$

$$\frac{2}{3} + \frac{2/3}{(3x-1)} = \frac{2x}{3x-1} = f(x)$$

so as $x \to \infty$ $f(x) = \frac{2}{3} + \cancelto{0}{\dfrac{2/3}{(3x-1)}}$

$f(x) = 2/3$

So horizontal asymptote at $y = 2/3$ for $f(x)$

- since improper, use long division:

$$3x-2 \overline{\smash{)}\,x} \quad \frac{1}{3}$$
$$-(x - 2/3)$$
$$2/3$$

So $\dfrac{x}{3x-2} = \dfrac{1}{3} + \dfrac{2/3}{3x-2} = f^{-1}(x)$

as $x \to \infty$ $\dfrac{2/3}{3x-2} \approx \dfrac{2/3}{3x} \approx 0$

So $f^{-1}(x) = \dfrac{1}{3} + \cancelto{0}{\dfrac{2/3}{3x-2}} = 1/3$

So horizontal asymptote at $y = 1/3$ for $f^{-1}(x)$

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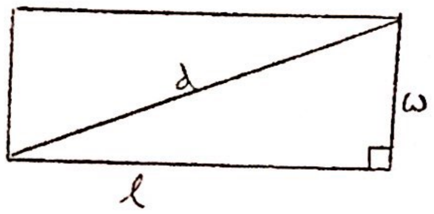
14)



$d = 10$ inches

$l = w + 2$

Pythagoras' theorem: $a^2 + b^2 = c^2$

$l^2 + w^2 = d^2$

$l = w+2$

$d = 10$

$(w+2)^2 + w^2 = 10^2$

$w^2 + 4w + 4 + w^2 = 10^2$

$2w^2 + 4w + 4 = 10^2$

$2w^2 + 4w - 96 = 0$

$w+2)(w+2)$
w^2+4w+4

$$w = \frac{-4 \pm \sqrt{4^2 - 4(2)(-96)}}{2(2)} = \frac{-4 \pm 28}{4}$$

$w = -\cancel{8}, 6$

dimensions of rectangle must have positive value

$w = 6$

$l = w + 2 = 8$

<u>Check</u> $6^2 + 8^2 = 10^2$

$100 = 100$ <u>OK</u>

<u>check</u> our length is 2 units more than our width (necessary requirement stated in problem)

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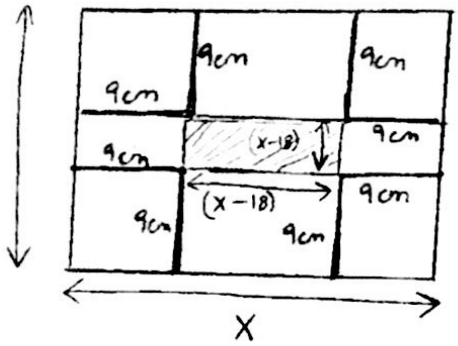
15)



shaded square
$(x-18)(x-18)$ = Area of base of box

the volume of the box = (Area of base) (height)

$$144 = (x-18)(x-18)(9\text{ cm})$$
$$144 = 9(x^2 - 36x + 324)$$
$$144 = 9x^2 - 324x + 2916$$
$$0 = 9x^2 - 324x + 2772$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{324 \pm \sqrt{5184}}{18}$$

$$x = \frac{324 \pm 72}{18} = 14, 22$$

$x \neq 14$ because the dimensions of the base of the box are $(x-18)(x-18)$

so $x = 22$ (must be!)

$x - 18 = 22 - 18 = 4$

<u>check</u> Volume of box $= 4 \times 4 \times 9 = 144\ cm^3$ <u>ok</u>

16) $N(t) = 1000e^{0.01t}$

t: hours
N: bacteria number

a) $t=0$

$N(t) = 1000e^{0.01(0)} = 1000e^{0} = 1000$

$e^{0}=1$

b) $N(t)=?$ when $t=4$ hours

$N(t) = 1000e^{0.01(4)} = 1040.81$

c) what value of t for $N(t)= 2(1000)=2000$

$2000 = 1000e^{0.01t}$

$2 = e^{0.01t}$

$\ln 2 = \ln e^{0.01t}$

$\ln 2 = 0.01t(\ln e)$

$\ln 2 = 0.01t$

$\frac{\ln 2}{0.01} = t$

$69.3 = t$

fun fact: $\ln e = 1$

So the amount of bacteria will double at 69.3 hours.

KKefalas