2.16 Write the Taylor's series expansion of the function $f(x) = \ln(x+n)$ about $x = 0$, where $n \neq 0$ is a known constant.

Solution

The Taylor's series for a function of one variable is given by Eq. (2.74):

$$f(x) = f(x_0) + (x - x_0)\left.\frac{df}{dx}\right|_{x = x_0} + \frac{(x-x_0)^2}{2!}\left.\frac{d^2f}{dx^2}\right|_{x = x_0} + \frac{(x-x_0)^3}{3!}\left.\frac{d^3f}{dx^3}\right|_{x = x_0} + \ldots + \frac{(x-x_0)^n}{n!}\left.\frac{d^nf}{dx^n}\right|_{x = x_0} + R_n(x)$$

In this problem, $x_0 = 0$, $f(x_0) = \ln(n)$, $\left.\frac{df}{dx}\right|_{x=0} = \left.\frac{1}{(x+n)}\right|_{x=0} = \frac{1}{n}$, $\left.\frac{d^2f}{dx^2}\right|_{x=0} = -\left.\frac{1}{(x-n)^2}\right|_{x=0} = -\frac{1}{n^2}$, $\left.\frac{d^3f}{dx^3}\right|_{x=0} = -\left.\frac{2}{(x-n)^3}\right|_{x=0} = \frac{2}{n^3}$, etc. Thus, the Taylor expansion of $f(x) = \ln(x+n)$ about $x = 0$ is:

$$\ln(x+n) = \ln(n) + \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} + \ldots$$

2.17 Write the Taylor's series expansion of the function $f(x, y) = e^{-x^2}\sin y$ about the point $(1, 3)$.

Solution

The Taylor's series for a function of two variables is given by Eq.(2.78):

$$f(x,y) = f(x_0, y_0) + \frac{1}{1!}\left[(x-x_0)\frac{\partial f}{\partial x}\bigg|_{x_0, y_0} + (y-y_0)\frac{\partial f}{\partial y}\bigg|_{x_0, y_0}\right] +$$

$$\frac{1}{2!}\left[(x-x_0)^2\frac{\partial^2 f}{\partial x^2}\bigg|_{x_0, y_0} + 2(x-x_0)(y-y_0)\frac{\partial^2 f}{\partial x \partial y}\bigg|_{x_0, y_0} + (y-y_0)^2\frac{\partial^2 f}{\partial y^2}\bigg|_{x_0, y_0}\right] +$$

$$+ \ldots + \frac{1}{n!}\left[\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(x-x_0)^k(y-y_0)^{n-k}\frac{\partial^n f}{\partial x^k \partial y^{n-k}}\bigg|_{x_0, y_0}\right]$$

In this problem, $x_0 = 1$ and $y_0 = 3$, $f(1, 3) = e^{-1}\sin 3$, $\frac{\partial f}{\partial x}\Big|_{x_0, y_0} = -2x(e^{-x^2}\sin y)\big|_{(1, 3)} = -2e^{-1}\sin 3$, $\frac{\partial f}{\partial y}\Big|_{x_0, y_0} = e^{-x^2}\cos y\big|_{(1, 3)} = e^{-1}\cos 3$, $\frac{\partial^2 f}{\partial x^2}\Big|_{x_0, y_0} = (4x^2e^{-x^2} - 2e^{-x^2})\sin y\big|_{(1, 3)} = 2e^{-1}\sin 3$, $\frac{\partial^2 f}{\partial y^2}\Big|_{x_0, y_0} = -e^{-x^2}\sin y\big|_{(1, 3)} = -e^{-1}\sin 3$, $\frac{\partial^2 f}{\partial x \partial y}\Big|_{x_0, y_0} = -2xe^{-x^2}\cos y\big|_{(1, 3)} = -2e^{-1}\cos 3$. Substituting into the Taylor's series yields:

$$e^{-x^2}\sin y = e^{-1}\sin 3 - 2(x-1)e^{-1}\sin 3 + (y-3)e^{-1}\cos 3$$

$$+ \frac{1}{2}[2(x-1)^2e^{-1}\sin 3 - 4(x-1)(y-3)e^{-1}\cos 3 - (y-3)^2e^{-1}\sin 3] + \ldots$$

3.3 The location $\bar{x}$ of the centroid of a circular sector is given by:

$$\bar{x} = \frac{2r\sin\theta}{3\theta}$$

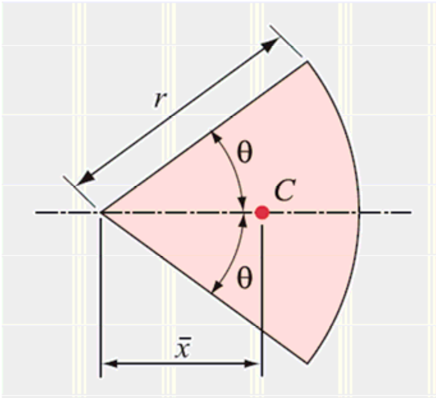
Determine the angle θ for which $\bar{x} = \frac{r}{2}$.

First, derive the equation that must be solved and then determine the root using the following methods:

(*a*) Use the bisection method. Start with $a = 1$ and $b = 2$, and carry out the first five iterations.

(*b*) Use the secant method. Start with the two points $x_1 = 1$ and $x_2 = 2$, and carry out the first five iterations.

(*c*) Use Newton's method. Start at $x_1 = 1$ and carry out the first five iterations.



Solution

Using 5 significant digits.

(*a*) $f(\theta) = \frac{2\sin\theta}{3\theta} - \frac{1}{2}$

$i = 1$, $a = 1$, $b = 2$, $f(1) = \frac{2\sin(1)}{3\cdot 1} - \frac{1}{2} = 0.06098$, $f(2) = \frac{2\sin(2)}{3\cdot 2} - \frac{1}{2} = -0.1969$,

$x_{NS1} = \frac{1+2}{2} = 1.5$, $f(1.5) = \frac{2\sin(1.5)}{3\cdot 1.5} - \frac{1}{2} = -0.056669$

$i = 2$, $a = 1$, $b = 1.5$, $x_{NS2} = \frac{1+1.5}{2} = 1.25$, $f(1.25) = \frac{2\sin(1.25)}{3\cdot 1.25} - \frac{1}{2} = 0.006125$

$i = 3$, $a = 1.25$, $b = 1.5$, $x_{NS3} = \frac{1.25+1.5}{2} = 1.375$, $f(1.375) = \frac{2\sin(1.375)}{3\cdot 1.375} - \frac{1}{2} = -0.24415$

$i = 4$, $a = 1.25$, $b = 1.375$, $x_{NS4} = \frac{1.25+1.375}{2} = 1.3125$,

$f(1.3125) = \frac{2\sin(1.3125)}{3\cdot 1.3125} - \frac{1}{2} = -0.0089135$

$i = 5$, $a = 1.25$, $b = 1.3125$, $x_{NS5} = \frac{1.25+1.3125}{2} = 1.2813$

(*b*) $\theta_{i+1} = \theta_i - \frac{f(\theta_i)(\theta_{i-1} - \theta_i)}{f(\theta_{i-1}) - f(\theta_i)}$

$\theta_1 = 1$, $\theta_2 = 2$, $f(\theta_1) = \frac{2\sin(1)}{3\cdot 1} - \frac{1}{2} = 0.06098$, , $f(\theta_2) = \frac{2\sin(2)}{3\cdot 2} - \frac{1}{2} = -0.1969$

Continued on next slide

Problem 3-3 continued

$$i = 2 \quad \theta_3 = 2 - \frac{(-0.1969)(1-2)}{0.06098 - -0.1969} = 1.2365, \quad f(\theta_3) = \frac{2\sin(1.2365)}{3\cdot 1.2365} - \frac{1}{2} = 0.0093093$$

$$i = 3 \quad \theta_4 = 1.2365 - \frac{0.0093093(2-1.2365)}{(-0.1969) - 0.0093093} = 1.271 \quad , \quad f(\theta_4) = \frac{2\sin(1.271)}{3\cdot 1.271} - \frac{1}{2} = 0.001126$$

$$i = 4 \quad \theta_5 = 1.271 - \frac{0.001126(1.2365 - 1.271)}{0.0093093 - 0.001126} = 1.2757, \quad f(\theta_5) = \frac{2\sin(1.2757)}{3\cdot 1.2757} - \frac{1}{2} = -4.5369\times 10^{-7}$$

$$i = 5 \quad \theta_6 = 1.2757 - \frac{(-4.5369\times 10^{-7})(1.271 - 1.2757)}{0.001126 - (-4.5369\times 10^{-7})} = 1.2757$$

$$(c) \quad f(\theta) = \frac{2\sin\theta}{3\theta} - \frac{1}{2}, \quad f'(\theta) = \frac{6\theta\cos\theta - 6\sin\theta}{9\theta^2}, \quad \theta_{i+1} = \theta_i - \frac{f(\theta_i)}{f'(\theta_i)} = \theta_i - \frac{(4\sin\theta_i - 3\theta_i)\theta_i}{4(\theta_i\cos\theta_i - \sin\theta_i)}$$

$$i = 1, \quad \theta_1 = 1, \quad \theta_2 = 1 - \frac{[4\sin(1) - (3\cdot 1)]1}{4(1\cos(1) - \sin(1))} = 1.3037$$

$$i = 2, \quad \theta_3 = 1.3037 - \frac{[4\sin(1.3037) - (3\cdot 1.3037)]1.3037}{4(1.3037\cos(1.3037) - \sin(1.3037))} = 1.2759$$

$$i = 3, \quad \theta_4 = 1.2759 - \frac{[4\sin(1.2759) - (3\cdot 1.2759)]1.2759}{4(1.2759\cos(1.2759) - \sin(1.2759))} = 1.2757$$

$$i = 4, \quad \theta_5 = 1.2757 - \frac{[4\sin(1.2757) - (3\cdot 1.2757)]1.2757}{4(1.2757\cos(1.2757) - \sin(1.2757))} = 1.2757$$

3.5 Determine the cube root of 155 by finding the numerical solution of the equation $x^3 - 155 = 0$. Use Newton's method. Start at $x = 155$ and carry out the first five iterations.

Solution

Using 5 significant digits.

$$f(x) = x^3 - 155, \quad f'(x) = 3x^2, \quad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^3 - 155}{3x_i^2}$$

$$i = 1, \quad x_1 = 155 \quad x_2 = 155 - \frac{155^3 - 155}{3 \cdot 155^2} = 103.34$$

$$i = 2, \quad x_3 = 103.34 - \frac{103.34^3 - 155}{3 \cdot 103.34^2} = 68.898$$

$$i = 3, \quad x_4 = 68.898 - \frac{68.898^3 - 155}{3 \cdot 68.898^2} = 45.943$$

$$i = 4, \quad x_5 = 45.943 - \frac{45.943^3 - 155}{3 \cdot 45.943^2} = 30.653$$

$$i = 5, \quad x_6 = 30.653 - \frac{30.653^3 - 155}{3 \cdot 30.653^2} = 20.490$$

3.9 The equation $f(x) = -x^{1/3} + 0.5x^2 - 2 = 0$ has a root between $x = 2$ and $x = 3$. To find the root by using the fixed-point iteration method, the equation has to be written in the form $x = g(x)$. Derive two possible forms for $g(x)$ — one by solving for x from the first term of the equation and the next by solving for x from the second term of the equation.
(*a*) Determine which form should be used according to the condition in Eq. (3.30).
(*b*) Carry out the first five iterations using both forms of $g(x)$ to confirm your determination in part (*a*).

Solution

(*a*)
Solving for x from the first term of the equation:

$x = (0.5x^2 - 2)^3$, $g(x) = (0.5x^2 - 2)^3$, $g'(x) = 3(0.5x^2 - 2)^2 x$, $g'(2.5) = 3(0.5 \cdot 2^2 - 2)^2 2 = 9.4922$
Since $|g'(2.5)| > 1$ the solution will not converge.

$x = (2x^{1/3} + 4)^{1/2}$, $g(x) = (2x^{1/3} + 4)^{1/2}$, $g'(x) = \dfrac{1}{3(2x^{1/3} + 4)^{1/2} x^{2/3}}$,

$g'(2) = \dfrac{1}{3(2 \cdot 2^{1/3} + 4)^{1/2} 2^{2/3}} = 0.0822$, $g'(3) = \dfrac{1}{3(2 \cdot 3^{1/3} + 4)^{1/2} 3^{2/3}} = 0.0611$

Since $|g'(2)| < 1$ and $|g'(3)| < 1$ the solution is expected to converge.
(*b*)
First five iterations using $x_{i+1} = (0.5x_i^2 - 2)^3$ starting with $x = 3$:

$i = 1$, $x_1 = 3$, $x_2 = (0.5 \cdot 3^2 - 2)^3 = 15.625$

$i = 2$, $x_3 = (0.5 \cdot 15.625^2 - 2)^3 = 1.731 \cdot 10^6$

$i = 3$, $x_4 = (0.5 \cdot (1.731 \cdot 10^6)^2 - 2)^3 = 3.3627 \cdot 10^{36}$

$i = 4$, $x_5 = (0.5 \cdot (3.3627 \cdot 10^{36})^2 - 2)^3 = 1.8073 \cdot 10^{218}$

First five iterations using $x_{i+1} = (2x_i^{1/3} + 4)^{1/2}$ starting with $x = 3$:

$i = 1$, $x_1 = 3$, $x_2 = (2 \cdot 3^{1/3} + 4)^{1/2} = 2.6238$

$i = 2$, $x_3 = (2 \cdot 2.6238^{1/3} + 4)^{1/2} = 2.5997$

$i = 3$, $x_4 = (2 \cdot 2.5997^{1/3} + 4)^{1/2} = 2.5981$

$i = 4$, $x_4 = (2 \cdot 2.5981^{1/3} + 4)^{1/2} = 2.5980$

The following problems are not in the textbook:

3.0 The function $f(x) = \dfrac{\sin(x/2)}{\sqrt{2+x}}$ has a maximum close to x=0. Find the coordinates of this maximum using the Secant method. Perform only three iterations, starting with x_1=0 and x_2=0.1. What is the relative error in your answer for x? Make sure that in your final answer you only show as many significant digits as allowed based on this relative error.

Solution:

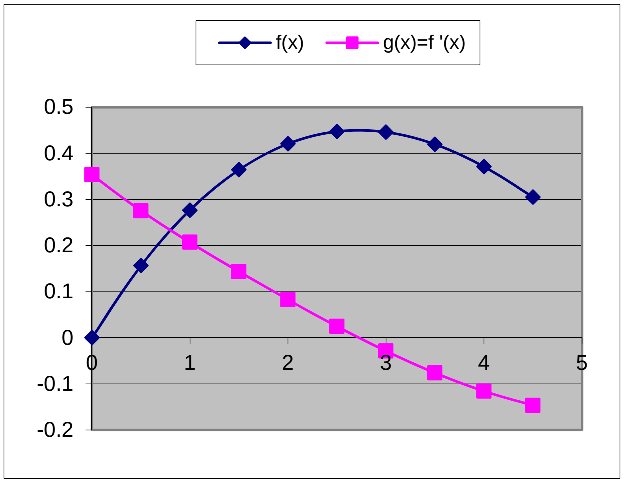
The maximum occurs when $f'(x) = 0$.

$$f'(x) = \frac{d}{dx}[\sin(0.5x) * (2+x)^{-0.5}] = 0.5 * \cos(0.5x) * (2+x)^{-0.5} - 0.5 * \sin(0.5x) * (2+x)^{-1.5} = 0$$

Note: Use radians!

incremental search and plot

	x	**f(x)**	**g(x)=f '(x)**
0.5	0	0	0.353553
	0.5	0.156472	0.275103
	1	0.276796	0.207204
	1.5	0.364351	0.143502
	2	0.420735	0.082484
	2.5	0.447356	0.024616
	3	0.446093	-0.02879
	3.5	0.419573	-0.07615
	4	0.371219	-0.11588
	4.5	0.305185	-0.14667



The secant method can be applied immediately:

Secant: $x_{i+1} = x_i - g(x_i) / g'(x_i)$, where $g'(x_i) = [g(x_i)-g(x_{i-1})]/(x_i-x_{i-1})$

i	**x_i**	**$g(x_i)$**	**$g'(x_i)$**	**x_{i+1}**	**$g(x_{i+1})$**	**ε=|x_{new}-x_{old}|/|x_{new}|**
0	0	0.353553		0.1	0.33639	
1	0.1	0.33639	-0.17163	2.05992	0.075362	0.951454
2	2.05992	0.075362	-0.13318	2.625771	0.010701	0.215499
3	2.625771	0.010701	-0.11427	2.719417	0.000538	0.034436

So g(x) has root at x =	2.719417		
or f(x) has maximum at x =	2.719417	with f(x) =	0.450098
and ε =	0.034436		
absolute error E = ε * x =	0.093646		

Final answer:	x =	2.72 ± 0.09	with f(x) =	0.45

3.00 Newton's method uses the equation $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$. However, if $f(x)$ has a multiple root {e.g. a triple root at $x = x_r$, so $f(x) = g(x)(x-x_r)^3$}, Newton's method converges very slowly to the root.

The Modified Newton's method converts the multiple root problem into a single root problem such that this method converges rapidly to the root.

A function $f(x)$ with a multiple root can be converted into another function $u(x)$ where the multiple root has become a single root if we apply Newton's method to $u(x)$, where $u(x) = \frac{f(x)}{f'(x)}$.

Assume a function $f(x)$ has a root with multiplicity n at $x = x_r$. Show how the Modified Newton's method in effect converts the "multiple root at $x = x_r$" problem into a "single root at $x = x_r$" problem. {You need to determine the equation for $u(x)$}

Solution:

The function $f(x)$ with a root at $x = x_r$ with multiplicity n can be written as follows:

$$f(x) = g(x)(x - x_r)^n$$

Taking the derivative we obtain: $f'(x) = g'(x)(x - x_r)^n - ng(x)(x - x_r)^{n-1}$

We can now obtain $u(x)$ as follows:

$$u(x) = \frac{f(x)}{f'(x)} = \frac{g(x)(x-x_r)^n}{g'(x)(x-x_r)^n - g(x)(x-x_r)^{n-1}} \quad \text{or} \quad u(x) = \frac{g(x)(x-x_r)}{g'(x)(x-x_r) - g(x)}$$

The equation for $u(x)$ has a single root at $x = x_r$. So, we have converted the multiple root problem into a single root problem.

<u>Find the equation for the modified Newton-Raphson method in terms of $f(x)$.</u>

The equation for the modified Newton-Raphson method in terms of $u(x)$ is $x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)}$.

$$u(x) = \frac{f(x)}{f'(x)} \quad \text{so} \quad u'(x) = \frac{[f'(x)]^2 - f(x)*f''(x)}{[f'(x)]^2}$$

$$\frac{u(x)}{u'(x)} = u(x) * \frac{1}{u'(x)} = \frac{f(x)}{f'(x)} * \frac{[f'(x)]^2}{[f'(x)]^2 - f(x)*f''(x)} = \frac{f(x)*f'(x)}{[f'(x)]^2 - f(x)*f''(x)}$$

So, for the Modified Newton-Raphson method $x_{i+1} = x_i - \frac{f(x_i)*f'(x_i)}{[f'(x_i)]^2 - f(x_i)*f''(x_i)}$